ALGEBRAIC SUBGROUPS OF THE PLANE CREMONA GROUP OVER A PERFECT FIELD

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ABSTRACT. We show that any infinite algebraic subgroup of the plane Cremona group over a perfect field is contained in a maximal algebraic subgroup of the plane Cremona group. We classify the maximal groups, and their subgroups of rational points, up to conjugacy by a birational map.

CONTENTS

1. INTRODUCTION

We study algebraic groups acting birationally and faithfully on a rational smooth projective surface over a perfect field $\mathbf{k}$. Any choice of birational map from that surface to the projective plane $\mathbb{P}^2$ induces an action of the algebraic group on $\mathbb{P}^2$ by birational transformations. Its subgroup of rational points can thus be viewed as a subgroup of the plane Cremona group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, which motivates the name *algebraic subgroup* of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$. The full classification - up to conjugacy - of algebraic subgroups of the plane Cremona group is open over many fields, because classifying the finite algebraic groups is very hard. Here is a selection of classification results over various perfect fields: [2, 6, 3, 14, 4, 15, 29, 36, 37]. The full classification of maximal algebraic subgroups of $\mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^2)$ (finite and infinite) can be found in [5] and the classification of the real locus of infinite algebraic subgroups of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ can be found in [30]. In this article, we restrict ourselves to consider infinite algebraic subgroups of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ over a perfect field $\mathbf{k}$ and we classify these groups up to conjugacy by elements of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ and up to inclusion. We also classify their subgroups of $\mathbf{k}$-rational points up to conjugation by elements of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ and up to inclusion. The two classifications are different as soon as $\mathbf{k}$ has a quadratic extension, see Corollary 1.3(2)–(3).

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Let us explain why we work over a perfect field. Given an algebraic subgroup G of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$, the strategy is to find a rational, regular and projective surface on which G acts by automorphisms and then use a G-equivariant Minimal Model Program to arrive on a conic fibration or a del Pezzo surface. It then remains to describe the automorphism group of that surface. Over a perfect field $\mathbf{k}$, regular implies smooth, and a smooth projective surface over $\mathbf{k}$ is a smooth projective surface over the algebraic closure $\overline{\mathbf{k}}$ of $\mathbf{k}$ equipped with an action of the Galois group $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ of $\overline{\mathbf{k}}$ over $\mathbf{k}$. In particular, the classification of rational smooth del Pezzo surfaces is simply the classification of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-actions on smooth del Pezzo surfaces over $\overline{\mathbf{k}}$ with $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-fixed points. This is straightforward if they have degree $\geqslant 6$, as we will see in §3 and §4. Over an imperfect field, regular does not imply smooth and a finite field extension may make appear singularities. The classification of regular del Pezzo surfaces is still open. In characteristic 2, there are regular, geometrically non-normal del Pezzo surfaces of degree 6 [16, Proposition 14.3, Proposition 14.5] and there are regular del Pezzo surfaces of degree 2 that are geometrically non-reduced [26, Proposition 3.4.1]. In particular, we cannot use directly the classification of regular del Pezzo surfaces over a separably closed field to describe the automorphism group of regular del Pezzo surfaces over an imperfect field, nor directly the classification of non-normal del Pezzo surfaces given in [28].

Now, assume again that $\mathbf{k}$ is a perfect field. Theorem 1.1, Theorem 1.2, Theorem 1.4 and Corollary 1.3 recover the classification results of [5] and [30] over $\mathbb{C}$ and $\mathbb{R}$ for infinite algebraic subgroups, and we will see that these results extend without any surprises over a perfect field with at least three elements. We leave it up to the reader to decide how surprising they find the results over the field with two elements.

By a theorem of Rosenlicht and Weil, for any algebraic subgroup G of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ there is a birational map $\mathbb{P}^2 \dashrightarrow X$ to a smooth projective surface X on which G acts by automorphisms, see Proposition 2.3. It conjugates G to a subgroup of $\mathrm{Aut}(X)$, the group scheme of automorphisms of X, and $G(\mathbf{k})$ is conjugate to a subgroup of $\mathrm{Aut}_{\mathbf{k}}(X)$. For a conic fibration $\pi\colon X \longrightarrow \mathbb{P}^1$ we denote by $\mathrm{Aut}(X,\pi) \subset \mathrm{Aut}(X)$ the subgroup preserving the conic fibration, by $\mathrm{Aut}(X/\pi) \subset \mathrm{Aut}(X,\pi)$ its subgroup inducing the identity on $\mathbb{P}^1$, and by $\mathrm{Aut}_{\mathbf{k}}(X,\pi)$ and $\mathrm{Aut}_{\mathbf{k}}(X/\pi)$ their $\mathbf{k}$-points. For a $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-invariant collection $p_1,\dots,p_r \in X(\overline{\mathbf{k}})$ of points, we denote by $\mathrm{Aut}_{\mathbf{k}}(X,p_1,\dots,p_r)$, resp. $\mathrm{Aut}_{\mathbf{k}}(X,\{p_1,\dots,p_r\})$, the subgroup of $\mathrm{Aut}_{\mathbf{k}}(X)$ fixing each p_i, resp. preserving the set $\{p_1,\dots,p_r\}$. A *splitting field* of $\{p_1,\dots,p_r\}$ is a finite normal extension $L/\mathbf{k}$ of smallest degree such that $p_1,\dots,p_r \in X(L)$ and such that $\{p_1,\dots,p_r\}$ is a union of $\mathrm{Gal}(L/\mathbf{k})$-orbits.

Suppose that $\mathbf{k}$ has a quadratic extension $L/\mathbf{k}$ and let g be the generator of $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathbb{Z}/2$. By $\mathcal{Q}^L$ we denote the $\mathbf{k}$-structure on $\mathbb{P}^1_L \times \mathbb{P}^1_L$ given by $(x,y)^g = (y^g,x^g)$. By $\mathcal{S}^{L,L'}$ we denote a surface obtained by blowing up $\mathcal{Q}^L$ in a point p of degree 2, where $L'/\mathbf{k}$ is the splitting field of p, whose geometric components are not on the same ruling of $\mathbb{P}^1_L \times \mathbb{P}^1_L$. We will show in Lemma 4.12 that its isomorphism class depends only on the isomorphism classes of L, L'. In Theorem 1.1(6b), we denote by $E \subset \mathcal{S}^{L,L'}$ its exceptional divisor.

Theorem 1.1. *Let* $\mathbf{k}$ *be a perfect field and* G *an infinite algebraic subgroup of* $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$. *Then there is a* $\mathbf{k}$*-birational map* $\mathbb{P}^2 \dashrightarrow X$ *that conjugates* G *to a subgroup of* $\mathrm{Aut}(X)$, *with* X *one of the following surfaces, where no indication of the* $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$*-action means the canonical action.*

(1) $X = \mathbb{P}^2$ *and* $\mathrm{Aut}(\mathbb{P}^2) \simeq \mathrm{PGL}_3$
(2) $X = \mathbb{F}_0$ *and* $\mathrm{Aut}(\mathbb{F}_0) \simeq \mathrm{Aut}(\mathbb{P}^1)^2 \rtimes \mathbb{Z}/2 \simeq \mathrm{PGL}_2^2 \rtimes \mathbb{Z}/2$

(3) $X=\mathcal{Q}^L$ *and* $\mathrm{Aut}(\mathcal{Q}^L)$ *is the* $\mathbf{k}$*-structure on* $\mathrm{Aut}(\mathbb{P}^1_L)^2\rtimes\mathbb{Z}/2$ *given by the* $\mathrm{Gal}(L/\mathbf{k})$*-action* $(A,B,\tau)^g=(B^g,A^g,\tau)$*, where* $L/\mathbf{k}$ *is a quadratic extension.*

(4) $X=\mathbb{F}_n$*,* $n\geqslant 2$*, and the action of* $\mathrm{Aut}(\mathbb{F}_n)$ *on* $\mathbb{P}^1$ *induces a split exact sequence*

$$1\longrightarrow V_{n+1}\longrightarrow \mathrm{Aut}(\mathbb{F}_n)\longrightarrow \mathrm{GL}_2/\mu_n\longrightarrow 1$$

where $\mu_n=\{a\,\mathrm{id}\mid a^n=1\}$ *and* V_{n+1} *is a vector space of dimension* $n+1$*.*

(5) X *is a del Pezzo surface of degree* 6 *with* $\mathrm{NS}(X_{\overline{\mathbf{k}}})^{\mathrm{Aut}_{\overline{\mathbf{k}}}(X)}=1$*. The action of* $\mathrm{Aut}_{\overline{\mathbf{k}}}(X)$ *on* $\mathrm{NS}(X_{\overline{\mathbf{k}}})$ *induces the split exact sequence*

$$1\to(\overline{\mathbf{k}}^*)^2\longrightarrow \mathrm{Aut}_{\overline{\mathbf{k}}}(X)\longrightarrow \mathrm{Sym}_3\times\mathbb{Z}/2\to 1.$$

Moreover, we are in one of the following cases.

(a) $\mathrm{rk}\,\mathrm{NS}(X)=1$ *and there is a quadratic extension* $L/\mathbf{k}$ *and a birational morphism* $\pi\colon X_L\longrightarrow\mathbb{P}^2_L$ *blowing up a point* $p=\{p_1,p_2,p_3\}$ *of degree* 3 *with splitting field* F *over* $\mathbf{k}$*, and one of the following cases holds:*

(i) $\mathrm{Gal}(F/\mathbf{k})\simeq\mathbb{Z}/3$ *and the action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1\to\mathrm{Aut}_L(\mathbb{P}^2,p_1,p_2,p_3)^{\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}}\longrightarrow\mathrm{Aut}_{\mathbf{k}}(X)\longrightarrow\mathbb{Z}/6\to 1$$

(ii) $\mathrm{Gal}(F/\mathbf{k})\simeq\mathrm{Sym}_3$ *and the action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1\to\mathrm{Aut}_L(\mathbb{P}^2,p_1,p_2,p_3)^{\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}}\longrightarrow\mathrm{Aut}_{\mathbf{k}}(X)\longrightarrow\mathbb{Z}/2\to 1,$$

(b) $\mathrm{rk}\,\mathrm{NS}(X)\geqslant 2$*,* $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)}=1$ *and* X *is one of the following:*

(i) X *is the blow-up of* $\mathbb{P}^2$ *in the coordinate points, and the action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1\to(\mathbf{k}^*)^2\longrightarrow\mathrm{Aut}_{\mathbf{k}}(X)\longrightarrow\mathrm{Sym}_3\times\mathbb{Z}/2\to 1.$$

(ii) X *is the blow-up of* $\mathbb{F}_0$ *in a point* $p=\{(p_1,p_1),(p_2,p_2)\}$ *of degree* 2*. The action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the exact sequence,*

$$1\to\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1,p_1,p_2)^2\longrightarrow\mathrm{Aut}_{\mathbf{k}}(X)\longrightarrow\mathrm{Sym}_3\times\mathbb{Z}/2\to 1$$

which is split if $\mathrm{char}(\mathbf{k})\neq 2$*.*

(iii) X *is the blow-up of* $\mathbb{P}^2$ *in a point* $p=\{p_1,p_2,p_3\}$ *of degree* 3 *with splitting field* L *such that* $\mathrm{Gal}(L/\mathbf{k})\simeq\mathbb{Z}/3$*. The action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1\to\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2,p_1,p_2,p_3)\longrightarrow\mathrm{Aut}_{\mathbf{k}}(X)\longrightarrow\mathbb{Z}/6\to 1$$

(iv) X *is the blow-up of* $\mathbb{P}^2$ *in a point* $p=\{p_1,p_2,p_3\}$ *of degree* 3 *with splitting field* L *such that* $\mathrm{Gal}(L/\mathbf{k})\simeq\mathrm{Sym}_3$*. The action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1\to\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2,p_1,p_2,p_3)\longrightarrow\mathrm{Aut}_{\mathbf{k}}(X)\longrightarrow\mathbb{Z}/2\to 1$$

where $\mathbb{Z}/2$ *is generated by a rotation.*

(c) $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)}=2$ *and there is a quadratic extension* $L/\mathbf{k}$ *and a birational morphism* $\nu\colon X\longrightarrow\mathcal{Q}^L$ *contracting two curves onto rational points* p_1,p_2 *or one curve onto a point* $\{p_1,p_2\}$ *of degree* 2 *with splitting field* $L'/\mathbf{k}$*. The action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1\to T^{L,L'}(\mathbf{k})\longrightarrow\mathrm{Aut}_{\mathbf{k}}(X)\longrightarrow\mathbb{Z}/2\times\mathbb{Z}/2\to 1$$

where $\nu \operatorname{Aut}_{\mathbf{k}}(X)\nu^{-1} = \operatorname{Aut}_{\mathbf{k}}(\mathcal{Q}^L, \{p_1,p_2\})$ *and* $T^{L,L'}$ *is the subgroup of* $\operatorname{Aut}_{\mathbf{k}}(\mathcal{Q}^L, p_1, p_2)$ *preserving the rulings of* $\mathcal{Q}_L^L$.

(6) $\pi\colon X \longrightarrow \mathbb{P}^1$ *is one of the following conic fibrations with*

$$\operatorname{rk}\operatorname{NS}(X_{\overline{\mathbf{k}}}/\mathbb{P}^1)^{\operatorname{Aut}_{\overline{\mathbf{k}}}(X,\pi)} = \operatorname{rk}\operatorname{NS}(X/\mathbb{P}^1)^{\operatorname{Aut}_{\mathbf{k}}(X,\pi)} = 1:$$

(a) $X/\mathbb{P}^1$ *is the blow-up of points* $p_1,\dots,p_r \in \mathbb{F}_n$, $n \geqslant 2$, *contained in a section* $S_n \subset \mathbb{F}_n$ *with* $S_n^2 = n$. *The geometric components of the* p_i *are on pairwise distinct geometric fibres and* $\sum_{i=1}^r \deg(p_i) = 2n$. *There are split exact sequences*

$$\begin{tikzcd}
 & (T_1/\mu_n) \rtimes \mathbb{Z}/2 \arrow[d, phantom, "\wr"] & \operatorname{Aut}(X) \arrow[d, phantom, "||"] & & \\
1 \arrow[r] & \operatorname{Aut}(X/\pi_X) \arrow[r] & \operatorname{Aut}(X,\pi_X) \arrow[r] & \operatorname{Aut}(\mathbb{P}^1,\Delta) \arrow[r] & 1 \\
1 \arrow[r] & \operatorname{Aut}_{\mathbf{k}}(X/\pi_X) \arrow[r] \arrow[d, phantom, "\wr"] & \operatorname{Aut}_{\mathbf{k}}(X,\pi_X) \arrow[r] \arrow[d, phantom, "||"] & \operatorname{Aut}_{\mathbf{k}}(\mathbb{P}^1,\Delta) \arrow[r] & 1 \\
 & (\mathbf{k}^*/\mu_n(\mathbf{k})) \rtimes \mathbb{Z}/2 & \operatorname{Aut}_{\mathbf{k}}(X) & &
\end{tikzcd}$$

where $\Delta = \pi(\{p_1,\dots,p_r\}) \subset \mathbb{P}^1$, T_1 *is the split one-dimensional torus and* μ_n *its subgroup of* n^{th} *roots of unity.*

(b) *There exist quadratic extensions* L *and* L' *of* $\mathbf{k}$ *such that* $X/\mathbb{P}^1$ *is the blow-up of* $\mathcal{S}^{L,L'}$ *in points* $p_1,\dots,p_r \in E$, $r \geqslant 1$. *The* p_i *are all of even degree, their geometric components are on pairwise distinct geometric components of smooth fibres and each geometric component of* E *contains half of the geometric components of each* p_i. *There are exact sequences*

$$\begin{tikzcd}
 & \mathrm{SO}^{L,L'} \rtimes \mathbb{Z}/2 \arrow[d, phantom, "\wr"] & \operatorname{Aut}(X) \arrow[d, phantom, "||"] & & \\
1 \arrow[r] & \operatorname{Aut}(X/\pi_X) \arrow[r] & \operatorname{Aut}(X,\pi_X) \arrow[r] & \operatorname{Aut}(\mathbb{P}^1,\Delta) \arrow[r] & 1 \\
1 \arrow[r] & \operatorname{Aut}_{\mathbf{k}}(X/\pi_X) \arrow[r] \arrow[d, phantom, "\wr"] & \operatorname{Aut}_{\mathbf{k}}(X,\pi_X) \arrow[r] \arrow[d, phantom, "||"] & (D_{\mathbf{k}}^{L,L'} \rtimes \mathbb{Z}/2) \cap \operatorname{Aut}_{\mathbf{k}}(\mathbb{P}^1,\Delta) \arrow[r] & 1 \\
 & \mathrm{SO}^{L,L'}(\mathbf{k}) \rtimes \mathbb{Z}/2 & \operatorname{Aut}_{\mathbf{k}}(X) & &
\end{tikzcd}$$

with $\Delta = \pi(\{p_1,\dots,p_r\}) \subset \mathbb{P}^1$ *and* $\mathrm{SO}^{L,L'} = \{(a,b) \in T^L \mid ab = 1\}$, *and*

- *if* L, L' *are* $\mathbf{k}$*-isomorphic, then* $\mathrm{SO}^{L,L'}(\mathbf{k}) \simeq \{a \in L^* \mid aa^g = 1\}$ *and* $D_{\mathbf{k}}^{L,L'} \simeq \{\alpha \in k^* \mid \alpha = \lambda\lambda^g, \lambda \in L\}$, *where* g *is the generator of* $\operatorname{Gal}(L/\mathbf{k})$,
- *if* L, L' *are not* $\mathbf{k}$*-isomorphic, then* $\mathrm{SO}^{L,L'}(\mathbf{k}) \simeq \mathbf{k}^*$ *and* $D_{\mathbf{k}}^{L,L'} \simeq \{\lambda\lambda^{gg'} \in F \mid \lambda \in K, \lambda\lambda^{g'} = 1\}$, *where* $\mathbf{k} \subset F \subset LL'$ *is the intermediate extension such that* $\operatorname{Gal}(F/\mathbf{k}) \simeq \langle gg' \rangle \subset \operatorname{Gal}(L/\mathbf{k}) \times \operatorname{Gal}(L'/\mathbf{k})$, *where* g, g' *are the generators of* $\operatorname{Gal}(L/\mathbf{k}), \operatorname{Gal}(L'/\mathbf{k})$, *respectively.*

We consider a family among (3), (5c), (5a), (5(b)ii), (5(b)iii), (5(b)iv), and (6b) empty if the point of requested degree or the requested field extension does not exist.

Theorem 1.1(5) is in fact the classification of rational del Pezzo surfaces of degree 6 over $\mathbf{k}$ up to isomorphism, and for any of the eight classes there is a field over which a surface in the class exists, see §4.

The next theorem lists the conjugacy classes in $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ of the groups in Theorem 1. Let G be an affine algebraic group and X/B a G-Mori fibre space (see Definition 2.11). We call it *G-birationally rigid* if for any G-equivariant birational map $\varphi\colon X \dashrightarrow X'$ to another G-Mori fibre space X'/B' we have $X' \simeq X$. In particular, $\varphi \operatorname{Aut}(X)\varphi^{-1} = \operatorname{Aut}(X')$. We call it *$G$-birationally superrigid* if any G-equivariant birational map $X \dashrightarrow X'$ to another G-Mori fibre space X'/B' is an isomorphism. If we replace G by $G(\mathbf{k})$ everywhere, we get the notion of $G(\mathbf{k})$-Mori fibre space, $G(\mathbf{k})$-birationally rigid and $G(\mathbf{k})$-birationally superrigid. The following theorem also shows that G-birationally (super)rigid does not imply $G(\mathbf{k})$-birationally (super)rigid.

The del Pezzo surfaces X and the conic fibrations $X/\mathbb{P}^1$ in Theorem 1.1 are $\operatorname{Aut}(X)$-Mori fibre spaces, and, except for the del Pezzo surfaces from (5c), they are $\operatorname{Aut}_{\mathbf{k}}(X)$-Mori fibre spaces.

Theorem 1.2. *Let $\mathbf{k}$ be a perfect field.*

1. *Any del Pezzo surface X and any conic fibration $X/\mathbb{P}^1$ from Theorem 1.1 is $\operatorname{Aut}(X)$-birationally superrigid.*
2. *Any del Pezzo surface X in Theorem 1.1(1)–(4), (5(b)ii)–(5(b)iv) and any conic fibration $X/\mathbb{P}^1$ from (6b) is $\operatorname{Aut}_{\mathbf{k}}(X)$-birationally superrigid.*
3. *Let X be a del Pezzo surface from Theorem 1.1(5a).*
   If $|\mathbf{k}| \geqslant 3$, then X is $\operatorname{Aut}_{\mathbf{k}}(X)$-birationally superrigid.
   If $|\mathbf{k}| = 2$, then there is an $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant birational map $X \dashrightarrow X'$, where X' is the del Pezzo surface from Theorem 1.1(5(b)ii).
4. *Let X be the del Pezzo surface from Theorem 1.1(5(b)i).*
   If $|\mathbf{k}| \geqslant 3$, then X is $\operatorname{Aut}_{\mathbf{k}}(X)$-birationally superrigid.
   If $|\mathbf{k}| = 2$, there are $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant birational maps $X \dashrightarrow \mathbb{F}_0$ and $X \dashrightarrow X'$, where X' is the del Pezzo surface of degree 6 from Theorem 1.1(5(b)ii).
5. *Any conic fibration $X/\mathbb{P}^1$ from Theorem 1.1(6a) is $\operatorname{Aut}_{\mathbf{k}}(X)$-birationally superrigid if $\mathbf{k}^*/\mu_n(\mathbf{k})$ is non-trivial. If $\mathbf{k}^*/\mu_n(\mathbf{k})$ is trivial and $X \dashrightarrow Y$ is an $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant birational map to a surface Y from Theorem 1.1, then $Y \simeq X$.*

We say that an algebraic subgroup G of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ is *maximal* if it is maximal with respect to inclusion among the algebraic subgroups of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$. We say that $G(\mathbf{k})$ is *maximal* if for any algebraic subgroup G' of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ containing $G(\mathbf{k})$, we have $G(\mathbf{k}) = G'(\mathbf{k})$.

By Theorem 1.2(4), if $|\mathbf{k}| = 2$ and X is a del Pezzo surface from (5(b)i), then $\operatorname{Aut}_{\mathbf{k}}(X)$ is not maximal: It is conjugate to a subgroup of $\operatorname{Aut}_{\mathbf{k}}(\mathbb{F}_0)$ and this inclusion is strict, because $\operatorname{Aut}_{\mathbf{k}}(X) \simeq \mathrm{Sym}_3 \times \mathbb{Z}/2$ has 12 elements, whereas $\operatorname{Aut}_{\mathbf{k}}(\mathbb{F}_0)$ has 72 elements. Similarly, $\operatorname{Aut}_{\mathbf{k}}(X)$ is not maximal if X is a del Pezzo surface from (5a) and $|\mathbf{k}| = 2$.

Corollary 1.3. *Let $\mathbf{k}$ be a perfect field and H an infinite algebraic subgroup of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$.*

1. *Then H is contained in a maximal algebraic subgroup G of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$.*
2. *Up to conjugation by a birational map, the maximal infinite algebraic subgroups of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ are precisely the groups $\operatorname{Aut}(X)$ in Theorem 1.1. Two maximal infinite subgroups $\operatorname{Aut}(X)$ and $\operatorname{Aut}(X')$ are conjugate by a biratonal map if and only if $X \simeq X'$.*
3. *$H(\mathbf{k})$ is maximal if and only if it is conjugate to one of the $\operatorname{Aut}_{\mathbf{k}}(X)$ from*
   - *(1)–(4), (5(b)ii)–(5(b)iv), (6),*
   - *(5a), (5(b)i) if $|\mathbf{k}| \geqslant 3$.*

   Two such groups $\operatorname{Aut}_{\mathbf{k}}(X)$ and $\operatorname{Aut}_{\mathbf{k}}(X')$ are conjugate by a birational map if and only if $X \simeq X'$.

Theorem 1.4. *Let $\mathbf{k}$ be a perfect field. The conjugacy classes of the maximal subgroups* $\mathrm{Aut}_{\mathbf{k}}(X)$ *of* $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ *from Theorem 1.1 are parametrised by*

- *(1), (2): one point*
- *(3): one point for each $\mathbf{k}$-isomorphism class of quadratic extensions of $\mathbf{k}$*
- *(4): one point for each $n \geqslant 2$*
- *(5(a)i) one point for any pair (L, F) of $\mathbf{k}$-isomorphism classes of quadratic extensions L and Galois extensions $F/\mathbf{k}$ with $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathbb{Z}/3$ if $|\mathbf{k}| \geqslant 3$*
- *(5(a)ii): one point for any pair (L, F) of $\mathbf{k}$-isomorphism classes of quadratic extensions L and Galois extensions $F/\mathbf{k}$ with $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3$*
- *(5(b)i): one point if $|\mathbf{k}| \geqslant 3$*
- *(5(b)ii): one point for each $\mathbf{k}$-isomorphism class of quadratic extensions of $\mathbf{k}$*
- *(5(b)iii): one point for each $\mathbf{k}$-isomorphism class of Galois extensions $F/\mathbf{k}$ with $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathbb{Z}/3$.*
- *(5(b)iv): one point for any $\mathbf{k}$-isomorphism class of Galois extensions $F/\mathbf{k}$ with $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3$.*
- *(6a): for each $n \geqslant 2$ the set of points $\{p_1, \dots, p_r\} \subset \mathbb{P}^1$ with $\sum_{i=1}^r \deg(p_i) = 2n$ up to the action of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1)$*
- *(6b): for each $n \geqslant 1$ and for each pair of $\mathbf{k}$-isomorphism classes of quadratic extensions (L, L'), the set of points $\{p_1, \dots, p_r\} \subset \mathbb{P}^1$ of even degree with $\sum_{i=1}^r \deg(p_i) = 2n$ up to the action of $D_{\mathbf{k}}^{L,L'}(\mathbf{k}) \rtimes \mathbb{Z}/2$*

We show the following consequence of [33] and [40, 39].

Proposition 1.5. *For any perfect field $\mathbf{k}$ there is a surjective homomorphism*

$$\Phi\colon \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) \longrightarrow \mathop{\ast}_J \bigoplus_I \mathbb{Z}/2,$$

where J is the set of points of degree 2 in $\mathbb{P}^2$ up to $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ and I is at least countable. If $[\overline{\mathbf{k}} : \mathbf{k}] = 2$, then $|I| = |\mathbf{k}|$.

If $\mathbf{k} = \mathbb{R}$ (or more generally $[\overline{\mathbf{k}} : \mathbf{k}] = 2$) then the abelianisation map of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is a homomorphism as in Proposition 1.5. By [30, Theorem 1.3] any infinite algebraic group acting on $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ that has non-trivial image in the abelianisation is a subgroup of the group in (6b), and this holds also if $[\overline{\mathbf{k}} : \mathbf{k}] = 2$. We will show a slightly more general statement over perfect fields with $[\overline{\mathbf{k}} : \mathbf{k}] > 2$, for which we need to introduce equivalence classes of Mori fibre spaces and links of type II.

We call two Mori fibre spaces $X_1/\mathbb{P}^1$ and $X_2/\mathbb{P}^1$ equivalent if there is a birational map $X_1 \dashrightarrow X_2$ that preserves the fibration. In particular, if $\varphi\colon X_1 \dashrightarrow X_2$ is a link of type II between Mori fibre spaces $X_1/\mathbb{P}^1$ and $X_2/\mathbb{P}^1$, then these two are equivalent. There is only one class of Mori fibre spaces birational to the Hirzebruch surface $\mathbb{F}_1$ [33, Lemma], because all rational points in $\mathbb{P}^2$ are equivalent up to $\mathrm{Aut}(\mathbb{P}^2)$. We denote by J_6 the set of classes of Mori fibre spaces birational to some $\mathcal{S}^{L,L'}$, and by J_5 the set of classes birational to a blow-up of $\mathbb{P}^2$ in a point of degree 4 whose geometric components are in general position. We call two Sarkisov links φ and φ' of type II between conic fibrations equivalent if the conic fibrations are equivalent and if the base-points of φ and φ' have the same degree. For a class C of equivalent rational Mori fibre spaces, we denote by $M(C)$ the set of equivalence classes of links of type II between conic fibrations in the class C whose base-points have degree $\geqslant 16$.

Theorem 1.6 ([33, Theorem 3, Theorem 4])**.** *For any perfect field with* $[\overline{\mathbf{k}}:\mathbf{k}]>2$ *there is a non-trivial homomorphism*

$$(*) \qquad \Psi\colon \operatorname{Bir}_{\mathbf{k}}(\mathbb{P}^2) \longrightarrow \bigoplus_{\chi\in M(\mathbb{F}_1)} \mathbb{Z}/2 * \Big(\mathop{\ast}_{C\in J_6} \bigoplus_{\chi\in M(C)} \mathbb{Z}/2\Big) * \Big(\mathop{\ast}_{C\in J_5} \bigoplus_{\chi\in M(C)} \mathbb{Z}/2\Big).$$

In fact, the homomorphism from Proposition 1.5 for $[\overline{\mathbf{k}}:\mathbf{k}]>2$ is induced by the one in Theorem 1.6.

We show that an infinite algebraic group acting birationally on $\mathbb{P}^2$ is killed by the homomorphism Ψ unless it is conjugate to a group of automorphisms acting on $\mathcal{S}^{L,L'}$ or a Hirzebruch surface.

Proposition 1.7. *Let* $\mathbf{k}$ *be a perfect field with* $[\overline{\mathbf{k}}:\mathbf{k}]>2$ *and let* Ψ *be the homomorphism* $(*)$*. Let* G *be an infinite algebraic subgroup of* $\operatorname{Bir}_{\mathbf{k}}(\mathbb{P}^2)$*. Then* $\Psi(G(\mathbf{k}))$ *is of order at most* 2 *and the following hold.*

(1) If $\Psi(G(\mathbf{k}))$ *is non-trivial, it is contained in the factor indexed by* $\mathbb{F}_1$ *or* $C\in J_6$ *and there is a* G*-equivariant birational map* $\mathbb{P}^2 \dashrightarrow X$ *that conjugates* G *to a subgroup of* $\operatorname{Aut}(X)$*, where* X *is as in Theorem 1(6a) or (6b), respectively.*

(2) Let $X/\mathbb{P}^1$ *be a conic fibration as in Theorem 1.1(6), which is the blow-up of* $\mathbb{F}_n$*,* $n\geqslant 2$*, or* $\mathcal{S}^{L,L'}$ *in points* $p_1,\dots,p_r$*. If* $\Psi(\operatorname{Aut}_{\mathbf{k}}(X))$ *is non-trivial, it is generated by the element whose non-zero entries are indexed by the* χ_i *that have* p_i *as base-point, where* $i\in\{1,\dots,r\}$ *is such that* $\deg(p_i)\geqslant 16$ *and* $|\{j\in\{1,\dots,r\}\mid \deg(p_j)=\deg(p_i)\}|$ *is odd.*

The analogous statement to Proposition 1.7 with the homomorphism from Proposition 1.5 for a perfect field $\mathbf{k}$ such that $[\overline{\mathbf{k}}:\mathbf{k}]=2$ can be found in [30].

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2. Surfaces and birational group actions

2.1. **Birational actions.** Throughout the article, $\mathbf{k}$ denotes a perfect field and $\overline{\mathbf{k}}$ an algebraic closure. By a surface X (or $X_{\mathbf{k}}$) we mean a smooth projective surface over $\mathbf{k}$ such that $X_{\overline{\mathbf{k}}} := X\times_{\operatorname{Spec}(\mathbf{k})}\operatorname{Spec}(\overline{\mathbf{k}})$ is irreducible. We denote by $X(\mathbf{k})$ the set of $\mathbf{k}$-rational points of X. The Galois group $\operatorname{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ acts on $X\times_{\operatorname{Spec}(\mathbf{k})}\operatorname{Spec}(\overline{\mathbf{k}})$ through the second factor. By a point of degree d we mean a $\operatorname{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbit $p=\{p_1,\dots,p_d\}\subset X(\overline{\mathbf{k}})$ of cardinality $d\geqslant 1$. The points of degree one are precisely the $\mathbf{k}$-rational points of X. Let $L/\mathbf{k}$ be an algebraic extension of $\mathbf{k}$ such that all p_i are L-rational points. By the blow up of p we mean the blow up of these d points, which is a morphism $\pi\colon X'\to X$ defined over $\mathbf{k}$, with exceptional divisor $E=E_1+\cdots+E_d$ where the E_i are disjoint (-1)-curves defined over L, and $E^2=-d$. We call E the exceptional divisor of p. More generally, a birational map $f\colon X\dashrightarrow X'$ is defined over $\mathbf{k}$ if and only if the birational map $f\times\mathrm{id}\colon X_{\overline{\mathbf{k}}}\dashrightarrow X'_{\overline{\mathbf{k}}}$ is $\operatorname{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-equivariant. In particular, $X\simeq X'$ if and only if there is a $\operatorname{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-equivariant isomorphism $X_{\overline{\mathbf{k}}}\longrightarrow X'_{\overline{\mathbf{k}}}$ (see also [8, §2.4]).

The surface X being projective and geometrically irreducible implies $\mathbf{k}[X_{\overline{\mathbf{k}}}]^*=(\overline{\mathbf{k}})^*$, so if $X(\mathbf{k})\neq\varnothing$ we have $\operatorname{Pic}(X_{\mathbf{k}})=\operatorname{Pic}(X_{\overline{\mathbf{k}}})^{\operatorname{Gal}(\overline{\mathbf{k}}/\mathbf{k})}$ [32, Lemma 6.3(iii)]. This holds in particular if X is $\mathbf{k}$-rational, because then it has a $\mathbf{k}$-rational point by the Lang-Nishimura theorem. Since numerical equivalence is $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-stable, also algebraic equivalence is, and hence $\mathrm{NS}(X_{\mathbf{k}}) = \mathrm{NS}(X_{\overline{\mathbf{k}}})^{\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})}$. The $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action on $\mathrm{NS}(X_{\overline{\mathbf{k}}})$ factors through a finite group, that is, its action factors through a finite group. Indeed, since $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ has only finite orbits on $\overline{\mathbf{k}}$, the orbit of any prime divisor of $X_{\overline{\mathbf{k}}}$ is finite. Then each generator of the finitely generated $\mathbb{Z}$-module $\mathrm{NS}(X_{\overline{\mathbf{k}}})$ has a finite $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbit, so the action of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ on the (finite) union of these orbits factors through a finite group.

If not mentioned otherwise, any surface, curve, point and rational map will be defined over the perfect field $\mathbf{k}$. By a geometric component of a curve C (resp. a point $p = \{p_1, \ldots, p_d\}$), we mean an irreducible component of $C_{\overline{\mathbf{k}}}$ (resp. one of $p_1, \ldots, p_d$).

By Châtelet's theorem, for $n \geqslant 1$ any smooth projective space X over $\mathbf{k}$ with $X(\mathbf{k}) \neq \varnothing$ such that $X_{\overline{\mathbf{k}}} \simeq \mathbb{P}^n_{\overline{\mathbf{k}}}$ is in fact already isomorphic to $\mathbb{P}^n$ over $\mathbf{k}$. This means in particular that $\mathbb{P}^2$ is the only rational del Pezzo surface of degree 9 and that a smooth curve of genus 0 with rational points is isomorphic to $\mathbb{P}^1$.

For a surface X, we denote by $\mathrm{Bir}_{\mathbf{k}}(X)$ its group of birational self-maps and by $\mathrm{Aut}_{\mathbf{k}}(X)$ the group of $\mathbf{k}$-automorphisms of X, which is the group of $\mathbf{k}$-rational points of a group scheme $\mathrm{Aut}(X)$ that is locally of finite type over $\mathbf{k}$ [10, Theorem 7.1.1] with at most countably many connected components.

An *algebraic group* G over a perfect field $\mathbf{k}$ is a (not necessarily connected) $\mathbf{k}$-group variety. In particular, G is reduced and hence smooth [10, Proposition 2.1.12]. We have $G_{\overline{\mathbf{k}}} = G \times_{\mathrm{Spec}(\mathbf{k})} \mathrm{Spec}(\overline{\mathbf{k}})$, on which $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ acts through the second factor. The definition of rational actions of algebraic groups on algebraic varieties goes back to Weil and Rosenlicht, see [35, 31].

Definition 2.1. We say that an algebraic group G *acts birationally* on a variety X if

(1) there are open dense subsets $U, V \subset G \times X$ and a birational map
$$G \times X \dashrightarrow G \times X, \quad (g, x) \mapstochar\dashrightarrow (g, \rho(g, x))$$
restricting to a isomorphism $U \to V$ and the projection of U and V to the first factor is surjective onto G, and

(2) $\rho(e, \cdot) = \mathrm{id}_X$ and $\rho(gh, x) = \rho(g, \rho(h, x))$ for any $g, h \in G$ and $x \in X$ such that $\rho(h, x), \rho(gh, x)$ and $\rho(g, \rho(h, x))$ are well defined.

The group $G(\mathbf{k})$ of $\mathbf{k}$-points of G is the subgroup of $G_{\overline{\mathbf{k}}}$ of elements fixed by the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action, so we have a map $G(\mathbf{k}) \longrightarrow \mathrm{Bir}_{\mathbf{k}}(X)$. Definition 2.1(2) implies that it is a homomorphism of groups, and Definition 2.1(1) is equivalent to the induced map $G(\mathbf{k}) \longrightarrow \mathrm{Bir}_{\mathbf{k}}(X)$, $g \longrightarrow f(g, \cdot)$ being a so-called morphism, see [7, Definition 2.1, Definition 2.2], usually denoted by $G \longrightarrow \mathrm{Bir}_{\mathbf{k}}(X)$ by abuse of notation. The notion of morphism from a variety to $\mathrm{Bir}_{\mathbf{k}}(X)$ goes back to M. Demazure [13] and J.-P. Serre [34].

We say that G is an *algebraic subgroup* of $\mathrm{Bir}_{\mathbf{k}}(X)$ if G acts birationally on X with trivial schematic kernel. We say that G *acts regularly* on X if the birational map in Definition 2.1(1) is an isomorphism. In that case, G is a subgroup of $\mathrm{Aut}(X)$ and we call X a *G-surface*.

Let G be an algebraic group acting birationally on surfaces X_1 and X_2 by birational maps $\rho_i \colon G \times X_i \dashrightarrow X_i$, $i = 1, 2$ as in Definition 2.1. A birational map $f \colon X_1 \dashrightarrow X_2$ is called G-*equivariant* if the following diagram commutes

$$\begin{tikzcd} G \times X_1 \arrow[r, dashed, "\rho_1"] \arrow[d, dashed, "\mathrm{id}_G \times f"'] & X_1 \arrow[d, dashed, "f"] \\ G \times X_2 \arrow[r, dashed, "\rho_2"] & X_2 \end{tikzcd}$$

In particular, if $\tilde{\rho}_i \colon G \longrightarrow \mathrm{Bir}_{\mathbf{k}}(X_i)$ denotes the induced morphism, the following diagram commutes

$$\begin{tikzcd} G(\mathbf{k}) \arrow[r, "\tilde{\rho}_1"] \arrow[dr, "\tilde{\rho}_2"'] & \mathrm{Bir}_{\mathbf{k}}(X_1) \arrow[d, "f \circ - \circ f^{-1}"] \\ & \mathrm{Bir}_{\mathbf{k}}(X_2) \end{tikzcd}$$

The following proposition is proven in [7, §2.6] over an algebraically closed field and its proof can be generalised over any perfect field.

Proposition 2.2 ([7, §2.6])**.** *Any algebraic subgroup of* $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ *is an affine algebraic group.*

The following proposition was proven separately by A. Weil and M. Rosenlicht [35, 31], but neither of them needed the new model to be smooth nor projective. Modern proofs can also be found in [25] over any field and in [12, 21] over algebraically closed fields.

Proposition 2.3. *Let X be a surface and G be an affine algebraic group acting birationally on X. Then there exists a G-surface Y and a G-equivariant birational map $X \dashrightarrow Y$. Furthermore, $G(\mathbf{k})$ has finite action on* $\mathrm{NS}(Y)$.

Proof. By [35, 31], there exists a normal not necessarily projective or smooth G-surface Y' and a G-equivariant birational map $X \dashrightarrow Y'$. The set Y'' of smooth points of Y' is G-stable, it is contained in a complete surface, which can be desingularised [24], so Y'' is quasi-projective. By [9, Corollary 2.14], Y'' has a G-equivariant completion Y'''. We now G-equivariantly desingularise Y''' to obtain the smooth projective surface Y [38, 23] (the sequence of blow-ups and normalisations over $\mathbf{k}$ can be done G-equivariantly).

The second claim is classical and for instance shown in [30, Lemma 2.10] over any perfect field. □

2.2. **Minimal surfaces.**

Definition 2.4. Let X be a surface, B a point or a smooth curve and $\pi\colon X \longrightarrow B$ a surjective morphism with connected fibres such that $-K_X$ is π-ample. We call $\pi\colon X \longrightarrow B$ a *rank r fibration*, where $r = \mathrm{rk}\,\mathrm{NS}(X/B)$.

- If $B = \mathrm{pt}$ is a point, the surface X is called *del Pezzo surface*. Then $X_{\overline{\mathbf{k}}}$ is isomorphic to $\mathbb{P}^1_{\overline{\mathbf{k}}} \times \mathbb{P}^1_{\overline{\mathbf{k}}}$ or to the blow-up of $\mathbb{P}^2_{\overline{\mathbf{k}}}$ in at most 8 points in general position. We call K_X^2 the *degree of* X. Note that $1 \leqslant K_X^2 \leqslant 9$.
- If B is a curve, then $\pi\colon X \longrightarrow B$ is called *conic fibration*; the general geometric fibre of π is isomorphic to $\mathbb{P}^1_{\overline{\mathbf{k}}}$ and a geometric singular fibre of π is the union of two secant (-1)-curves over $\overline{\mathbf{k}}$. Moreover, if X is rational, then $B = \mathbb{P}^1$, see for instance [33, Lemma 2.4].
- If $r = 1$, then $\pi\colon X \longrightarrow B$ is called *Mori fibre space*.

We may write X/B instead of $\pi\colon X \longrightarrow B$. Let X/B and X'/B' be conic fibrations. We say that a birational map $\varphi\colon X \dashrightarrow X'$ *preserves the fibration* or is a *birational map of conic fibrations* if the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi"] \arrow[d] & X' \arrow[d] \\ B \arrow[r, "\simeq"] & B' \end{tikzcd}$$

commutes.

For a surface X, we can run the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-equivariant Minimal Model program on $X_{\overline{\mathbf{k}}}$, because the action of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ on $\mathrm{NS}(X_{\overline{\mathbf{k}}})$ is finite. The end result is a $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-Mori fibre space $Y_{\overline{\mathbf{k}}}/B_{\overline{\mathbf{k}}}$ as in Definition 2.4, which is equivalent to Y/B being a Mori fibre space.

Example 2.5.

(1) For $n \geqslant 0$, the Hirzebruch surface $\mathbb{F}_n$ is the quotient of the action of $(\mathbb{G}_m)^2$ on $(\mathbb{A}^2\backslash\{0\})^2$ by

$$(\mathbb{G}_m)^2 \times (\mathbb{A}^2\backslash\{0\})^2 \longrightarrow (\mathbb{A}^2\backslash\{0\})^2,\ (\mu,\rho),(y_0,y_1,z_0,z_1) \mapsto (\mu\rho^{-n}y_0, \mu y_1, \rho z_0, \rho z_1).$$

The class of (y_0,y_1,z_0,z_1) is denoted by $[y_0 : y_1; z_0 : z_1]$. The projection $\pi_n\colon \mathbb{F}_n \longrightarrow \mathbb{P}^1$ given by $[y_0 : y_1; z_0 : z_1] \mapsto [z_0 : z_1]$ is a conic fibration and the special section $S_{-n} \subset \mathbb{F}_n$ is given by $y_0 = 0$.

(2) Let p and p' be two points of degree 2 in $\mathbb{P}^2$ with splitting field $L/\mathbf{k}$ and $L'/\mathbf{k}$, respectively, such that their geometric components are in general position. We denote by $\mathcal{S}^{L,L'}$ a del Pezzo surface obtained by first blowing up p, p', and then contracting the line passing through one of the two points. It has a natural conic fibration structure $\mathcal{S}^{L,L'} \longrightarrow \mathbb{P}^1$; the fibres are the strict transforms of the conics in $\mathbb{P}^2$ passing through the two points.

Lemma 2.6. [33, Lemma 6.11] *Let $L/\mathbf{k}$ be a finite extension. Let $p_1,\dots,p_4,q_1,\dots,q_4 \in \mathbb{P}^2(L)$ such that the sets $\{p_1,\dots,p_4\}$ and $\{q_1,\dots,q_4\}$ are $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-invariant and no three of the p_i and no three of the q_i are collinear. Suppose that for any $g \in \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ there exists $\sigma \in \mathrm{Sym}_4$ such that $p_i^g = p_{\sigma(i)}$ and $q_i^g = q_{\sigma(i)}$ for $i = 1,\dots,4$. Then there exists $\alpha \in \mathrm{PGL}_3(\mathbf{k})$ such that $\alpha(p_i) = q_i$ for $i = 1,\dots,4$.*

Remark 2.7. The argument of [33, Lemma 6.11] can be applied to show the following analogue of Lemma 2.6 on $\mathbb{P}^1$: let $F/\mathbf{k}$ be a finite extension and $p_1,p_2,p_3,q_1,q_2,q_3 \in \mathbb{P}^1(F)$ such that the sets $\{p_1,p_3,p_3\}$ and $\{q_1,q_2,q_3\}$ are $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-invariant. Suppose that for any $g \in \mathrm{Gal}(F/\mathbf{k})$ there exists $\sigma \in \mathrm{Sym}_3$ such that $p_i^g = p_{\sigma(i)}$ and $q_i^g = q_{\sigma(i)}$ for $i = 1,2,3$. Then there exists $\alpha \in \mathrm{PGL}_2(\mathbf{k})$ such that $\alpha(p_i) = q_i$ for $i = 1,2,3$.

Lemma 2.8. [33, Remark 6.1, Lemma 6.13] *Let $\pi\colon X \longrightarrow \mathbb{P}^1$ be a Mori fibre space and suppose that X is rational. Then X is isomorphic to a Hirzebruch surface, to a del Pezzo surface $\mathcal{S}^{L,L'}$ or to a del Pezzo surface obtained by blowing up a point of degree 4 in $\mathbb{P}^2$.*

Proposition 2.9. *Let X/B be a Mori fibre space. If B is a point, then X is rational if and only if $K_X^2 \geqslant 5$ and $X(\mathbf{k}) \neq \varnothing$.*

Proof. Suppose that $d := K_X^2 \geqslant 5$ and that $X(\mathbf{k})$ contains a point r. If $d = 7$, then $X_{\overline{\mathbf{k}}}$ contains three (-1)-curves, one of which must be $\mathbf{k}$-rational, contradicting $\mathrm{rk}\,\mathrm{NS}(X) = 1$. If $d = 8$, the blow-up of r is a del Pezzo surface of degree 7, which has two disjoint

(-1)-curves over $\overline{\mathbf{k}}$ that are either both $\mathbf{k}$-rational or they make up a $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbit of curves. Contracting them induces a birational map over $\mathbf{k}$ to a del Pezzo surface of degree 9 with a rational point, which hence is $\mathbb{P}^2$. This argument also holds if $\mathrm{rk}\,\mathrm{NS}(X) = 2$. Let $d = 6$. If r is contained in a curve of negative self-intersection, then that curve is a $\mathbf{k}$-rational (-1)-curve, contradicting $\mathrm{rk}\,\mathrm{NS}(X) = 1$. If r is not contained in any curve of negative self-intersection, the blow-up of r contains a curve with three pairwise disjoint geometric components of self-intersection -1. Their contraction yields a birational map $X \dashrightarrow Y$, where Y is a del Pezzo surface of degree 8 with a rational point, so Y is rational by the argument above. If $d = 5$, then again $\mathrm{rk}\,\mathrm{NS}(X) = 1$ implies that r is not in a (-1)-curve. After blowing up r we can contract a curve with five pairwise disjoint geometric components and arrive on a del Pezzo surface of degree 9, which is $\mathbb{P}^2$ because it has a rational point.

Let's prove the converse implication. If X is a rational del Pezzo surface, then $X(\mathbf{k}) \neq \varnothing$ by the Lang-Nishimura theorem. The remaining claim follows from the classification of *Sarkisov links* (see definition in Section 7.1) between rational Mori fibre spaces over a perfect field [19, Theorem 2.6]. Indeed, any birational map between del Pezzo surfaces over $\mathbf{k}$ with Picard rank 1 decomposes into Sarkisov links and automorphisms [19, Theorem 2.5]. The list of Sarkisov links implies the following: for a del Pezzo surface X with $\mathrm{rk}\,\mathrm{NS}(X) = 1$ and $K_X^2 \leqslant 4$, any Sarkisov link $X \dashrightarrow Y$ that is not an isomorphism is to a del Pezzo surface Y, either of degree $K_Y^2 \leqslant 4$ and $\mathrm{rk}\,\mathrm{NS}(Y) = 1$, or of degree $K_Y^2 = 3$ and Y carries moreover the structure $Y \longrightarrow \mathbb{P}^1$ of a Mori fibre space. From the latter, any Sarkisov link $Y \dashrightarrow Z$ is to a del Pezzo surface Z of degree $\leqslant 4$, either with $\mathrm{rk}\,\mathrm{NS}(Z) = 1$ or it preserves the fibration and $\mathrm{rk}\,\mathrm{NS}(Z) = 2$. In particular, X cannot be joined to $\mathbb{P}^2$ by a birational map. □

Lemma 2.10. *If X is a del Pezzo surface of degree $K_X^2 \leqslant 5$, then $\mathrm{Aut}_{\overline{\mathbf{k}}}(X)$ is finite.*

Proof. It suffices to show the claim for $\mathbf{k} = \overline{\mathbf{k}}$. Then X is the blow-up of $p_1, \ldots, p_r \in \mathbb{P}^2$ in general position with $r = 9 - K_X^2 \geqslant 4$. It has finitely many (-1)-curves, say n of them, and the action of $\mathrm{Aut}_{\mathbf{k}}(X)$ on the set of the (-1)-curves induces an exact sequence

$$1 \to \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, \ldots, p_r) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathrm{Sym}_n\,.$$

Since $p_1, \ldots, p_r$ are in general position and $r \geqslant 4$, the group $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, \ldots, p_r)$ is trivial, which yields the claim. □

2.3. **Relatively minimal surfaces.** We now generalise the notion of being a minimal surface to being minimal relative to the action of an affine algebraic group.

Definition 2.11. Let G be an affine algebraic group, let X be a G-surface and $\pi\colon X \longrightarrow B$ a rank r fibration.

(1) If π is G-equivariant and $r' := \mathrm{rk}\,\mathrm{NS}(X_{\overline{\mathbf{k}}}/B_{\overline{\mathbf{k}}})^{G_{\overline{\mathbf{k}}} \times \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})}$, we call π a *G-equivariant rank r' fibration.* If $r' = 1$ we call it a *G-Mori fibre space.*
(2) If π is $G(\mathbf{k})$-equivariant and $r'' := \mathrm{rk}\,\mathrm{NS}(X/B)^{G(\mathbf{k})}$, we call π a *G-equivariant rank r'' fibration.* If $r'' = 1$ we call it *$G(\mathbf{k})$-Mori fibre space.*

If a rank r fibration $X \longrightarrow B$ is G-equivariant, we have $r \geqslant r'' \geqslant r'$. A G-Mori fibre space is not necessarily a $G(\mathbf{k})$-Mori fibre space, since $G(\mathbf{k})$-equivariant does not imply G-equivariant. Examples are, for instance, the del Pezzo surfaces in Lemma 4.11 and Lemma 4.9 (see also Theorem 1.1(5c)), that are $\mathrm{Aut}(X)$-Mori fibre spaces but not $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre spaces.

If G is connected, Blanchard's Lemma [9, Theorem 7.2.1] implies that a G-Mori fibre space is a Mori fibre space. However, the affine algebraic groups we are going to work with are not necessarily connected. All del Pezzo surfaces X of degree 6 in §4 are $\mathrm{Aut}(X)$-Mori fibre spaces, all but two of them are also $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre spaces and only two of them are Mori fibre spaces.

Let G be an affine algebraic group and X a G-surface. The action $\rho\colon G\times X\longrightarrow X$ from Definition 2.1 being defined over $\mathbf{k}$ is equivalent to $\bar\rho:=\rho\times\mathrm{id}\colon G_{\overline{\mathbf{k}}}\times X_{\overline{\mathbf{k}}}\longrightarrow X_{\overline{\mathbf{k}}}$ being $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-equivariant, *i.e.* $\bar\rho(g,x)^h=\bar\rho(g^h,x^h)$ for any $h\in\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$, $g\in G_{\overline{\mathbf{k}}}$, $x\in X_{\overline{\mathbf{k}}}$. We can therefore see the G-action on X as the $(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})\times G_{\overline{\mathbf{k}}})$-action on $X_{\overline{\mathbf{k}}}$

$$(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})\times G_{\overline{\mathbf{k}}})\times X_{\overline{\mathbf{k}}}\longrightarrow X_{\overline{\mathbf{k}}},\quad (h,g,x)\mapsto\bar\rho(g^h,x^h)$$

satisfying $\bar\rho(g^h,x^h)=\bar\rho(g,x)^h$ for any $h\in\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$, $g\in G_{\overline{\mathbf{k}}}$, $x\in X_{\overline{\mathbf{k}}}$.

Remark 2.12. Let G be an affine algebraic group and X a G-surface such that $X_{\overline{\mathbf{k}}}$ is rational. By Proposition 2.3, the group $G_{\overline{\mathbf{k}}}$ and hence also the group $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})\times G_{\overline{\mathbf{k}}}$ has finite action on $\mathrm{NS}(X_{\overline{\mathbf{k}}})$. We can run the $(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})\times G_{\overline{\mathbf{k}}})$-equivariant Minimal Model program on $X_{\overline{\mathbf{k}}}$, and by [20, Example 2.18] the end result is a G-Mori fibre space Y/B. We then restrict to the $G(\mathbf{k})$-action on Y and recall that $G(\mathbf{k})$ has finite action on $\mathrm{NS}(Y)$ by Proposition 2.3. Since Y/B is G-equivariant, it is also $G(\mathbf{k})$-equivariant, and we can run the $G(\mathbf{k})$-equivariant Minimal Model Program on Y, whose end result is then a $G(\mathbf{k})$-Mori fibre space.

Let us tidy up the direction for classifying the infinite algebraic subgroups of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$.

Proposition 2.13. *Let G be an infinite algebraic subgroup of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$. Then there exists a G-equivariant birational map $\mathbb{P}^2\dashrightarrow X$ to a G-Mori fibre space X/B that is one of the following:*

(1) B is a point and $X\simeq\mathbb{P}^2$ or X is a del Pezzo surface of degree 6 or 8.
(2) $B=\mathbb{P}^1$ and there exists a birational morphism of conic fibrations $X\longrightarrow\mathcal{S}^{L,L'}$ or $X\longrightarrow\mathbb{F}_n$ for some $n\geqslant 0$.

Proof. By Proposition 2.2, G is an affine algebraic group. By Proposition 2.3, there is a G-surface X' and a G-equivariant birational map $\phi\colon\mathbb{P}^2\dashrightarrow X'$. We now apply the $(G_{\overline{\mathbf{k}}}\times\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))$-equivariant Minimal Model Program and obtain a G-equivariant birational morphism $X'\longrightarrow X$ to a G-Mori fibre space $\pi\colon X\longrightarrow B$, see Remark 2.12.

If B is a point, then X is a del Pezzo surface. Since G is infinite, Lemma 2.10 implies that $K_X^2\geqslant 6$. If $K^2=7$, then $X_{\overline{\mathbf{k}}}$ contains exactly three (-1)-curves, one of which is $G_{\overline{\mathbf{k}}}\times\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-invariant, so X is not a G-Mori fibre space. It follows that $K_X^2\in\{6,8,9\}$, and if $K_X^2=9$, then $X\simeq\mathbb{P}^2$ by Châtelet's Theorem.

Suppose that $B=\mathbb{P}^1$. Then there is a birational morphism $X\longrightarrow Y$ of conic fibrations onto a Mori fibre space $Y/\mathbb{P}^1$. By Lemma 2.8, Y is a Hirzebruch surface, $Y\simeq\mathcal{S}$ or Y is the blow-up of $\mathbb{P}^2$ in a point of degree 4 whose geometric components are in general position. The latter is a del Pezzo surface of degree 5, so by Lemma 2.10 the group $\mathrm{Aut}_{\overline{\mathbf{k}}}(Y)$ is finite, which does not occur under our hypothesis. It follows that $Y\simeq\mathbb{F}_n$, $n\geqslant 0$, or $Y\simeq\mathcal{S}^{L,L'}$. □

Lemma 2.14.

(1) If X is a del Pezzo surface, then $\mathrm{Aut}(X)$ is an affine algebraic group.
(2) Let $\pi\colon X\to\mathbb{P}^1$ be a conic fibration such that $X_{\overline{\mathbf{k}}}$ is rational. Then $\mathrm{Aut}(X,\pi)$ is an affine algebraic group.

Proof. (1) Let $N := h^0(-K_X)$. Then $\mathrm{Aut}(X)$ preserves the ample divisor $-K_X$, thus it is conjugate via the embedding $|-K_X|\colon X \hookrightarrow \mathbb{P}^{N-1}$ to a closed subgroup of $\mathrm{Aut}(\mathbb{P}^{N-1}) \simeq \mathrm{PGL}_N$ and is hence affine.

(2) Let G be the schematic kernel of $\mathrm{Aut}(X,\pi) \longrightarrow \mathrm{Aut}(\mathrm{NS}(X))$. If D is an ample divisor on X, it is fixed by G and hence (as above) G is an affine algebraic group. Since $X_{\overline{\mathbf{k}}}$ is rational and has the structure of a conic fibration, we have $\mathrm{NS}(X) \simeq \mathbb{Z}^n$ for some $n \geqslant 2$, and it is generated by $-K_X$, the general fibre and components of the singular fibres. The (abstract) group $H := \mathrm{Aut}(X,\pi)/G$ acts faithfully on $\mathrm{NS}(X)$, fixes $-K_X$ and the general fibre and permutes the components of the singular fibres. It follows that H is isomorphic (as abstract group) to a subgroup of $\mathrm{GL}_n(\mathbb{Z})$ whose elements have entries in $\{0, \pm 1\}$. Therefore, H is finite and hence $\mathrm{Aut}(X,\pi)$ is an affine algebraic group. □

In particular, if X is a del Pezzo surface, the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action on $\mathrm{Aut}_{\overline{\mathbf{k}}}(X)$ is a $\mathbf{k}$-structure with fixed locus $\mathrm{Aut}_{\mathbf{k}}(X)$. Similarly, if $\pi\colon X \to \mathbb{P}^1$ is a conic fibration such that $X_{\overline{\mathbf{k}}}$ is rational, then the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action on $\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\pi)$ is a $\mathbf{k}$-structure with fixed locus $\mathrm{Aut}_{\mathbf{k}}(X,\pi)$.

Our goal is to classify algebraic subgroups of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ up to conjugacy and inclusion. Proposition 2.13 and Lemma 2.14 imply that it suffices to classify up to conjugacy and inclusion the automorphism groups of del Pezzo surfaces of degree 6 and 8 and the automorphism groups of certain conic fibrations.

3. Del Pezzo surfaces of degree 8

We now classify the rational del Pezzo surfaces of degree 8. Over an algebraically closed field, any such surface is isomorphic to the blow-up of $\mathbb{P}^2$ in a point or to $\mathbb{P}^1 \times \mathbb{P}^1$. Over $\mathbb{R}$, there are exactly two rational models of the latter, namely the quadric surfaces given by $w^2+x^2-y^2-z^2=0$ or $w^2+x^2+y^2-z^2=0$ in $\mathbb{P}^3$. The first is isomorphic to $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ and the second is the $\mathbb{R}$-form of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ given by $(x,y) \mapsto (y^g, x^g)$, where $\langle g\rangle = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We now show that the classification is similar over an arbitrary perfect field $\mathbf{k}$.

Definition 3.1. Suppose that $\mathbf{k}$ has a quadratic extension $L/\mathbf{k}$. We denote by $\mathcal{Q}^L$ the $\mathbf{k}$-structure on $\mathbb{P}^1_L \times \mathbb{P}^1_L$ given by $([u_0:u_1],[v_0:v_1]) \mapsto ([v_0^g:v_1^g],[u_0^g:u_1^g])$, where g is the generator of $\mathrm{Gal}(L/\mathbf{k})$.

The surface $\mathcal{Q}^L$ is a del Pezzo surface of degree 8 and it is rational by Proposition 2.9 because the point $([1:1],[1:1]) \in \mathcal{Q}^L(\mathbf{k})$.

Lemma 3.2. *Let X be a rational del Pezzo surface of degree 8.*

1. *We have $\mathrm{rk}\,\mathrm{NS}(X)=2$ if and only if $X \simeq \mathbb{F}_0$ or $X \simeq \mathbb{F}_1$, and $\mathrm{rk}\,\mathrm{NS}(X)=1$ if and only if $X \simeq \mathcal{Q}^L$ for some quadratic extension $L/\mathbf{k}$.*
2. *$X \simeq \mathcal{Q}^L$ if and only if for any $p \in X(\mathbf{k})$ there is a birational map $X \dashrightarrow \mathbb{P}^2$ that is the composition of the blow-up of p and the contraction of a curve onto a point of degree 2 in $\mathbb{P}^2$ whose splitting field is L.*
3. *We have $\mathcal{Q}^L \simeq \mathcal{Q}^{L'}$ if and only if L and L' are $\mathbf{k}$-isomorphic.*

Proof. (1–2) The surface $X_{\overline{\mathbf{k}}}$ is a del Pezzo surface of degree 8 over $\overline{\mathbf{k}}$ and is hence isomorphic to $\mathbb{P}^1_{\overline{\mathbf{k}}} \times \mathbb{P}^1_{\overline{\mathbf{k}}}$ or to $(\mathbb{F}_1)_{\overline{\mathbf{k}}}$. In the latter case, the unique (-1)-curve is $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-invariant, hence $X \simeq \mathbb{F}_1$. Suppose that $X_{\overline{\mathbf{k}}}$ is isomorphic to $\mathbb{P}^1_{\overline{\mathbf{k}}} \times \mathbb{P}^1_{\overline{\mathbf{k}}}$ and consider the blow-up $\pi_1\colon Y \longrightarrow X$ of X in a rational point $p \in X(\mathbf{k})$ (such a point exists by Proposition 2.9). Then Y is a del Pezzo surface of degree 7 and $Y_{\overline{\mathbf{k}}}$ has three (-1)-curves, one of which is the exceptional divisor over the rational point p. The union of the other two

(-1)-curves $C_1, C_2 \subset Y_{\overline{\mathbf{k}}}$ is preserved by $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$, and hence their contraction yields a birational morphism $\pi_2 \colon Y \longrightarrow \mathbb{P}^2$. If each of C_1 and C_2 is preserved by $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$, then $\varphi := \pi_1\pi_2^{-1} \colon \mathbb{P}^2 \dashrightarrow X$ has two rational base-points. The pencil of lines through each base-point is sent onto a fibration of X, and Lemma 2.8 implies that X is a Hirzebruch surface, so $X \simeq \mathbb{F}_0$. If $C_1 \cup C_2$ is a $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbit of curves, then φ has a base-point q of degree 2. By Remark 2.6 we can assume that q is of the form $q = \{[a_1 : 1 : 0], [a_2 : 1 : 0]\}$, $a_1, a_2 \in \overline{\mathbf{k}}$. We consider the projection $\psi \colon \mathbb{P}^2_{\overline{\mathbf{k}}} \dashrightarrow \mathbb{P}^1_{\overline{\mathbf{k}}} \times \mathbb{P}^1_{\overline{\mathbf{k}}}$ away from q

$$\psi \colon [x : y : z] \mapsto ([x - a_1 y : z], [x - a_2 y : z])$$

$$\psi^{-1} \colon ([u_0 : u_1], [v_0 : v_1]) \mapsto [-a_2 u_0 v_1 + a_1 v_0 u_1 : -u_0 v_1 + v_0 u_1 : (a_1 - a_2) u_1 v_1]$$

whose inverse ψ^{-1} has base-point $([1 : 1], [1 : 1])$. There exists an isomorphism $\alpha \colon X_{\overline{\mathbf{k}}} \xrightarrow{\simeq} \mathbb{P}^1_{\overline{\mathbf{k}}} \times \mathbb{P}^1_{\overline{\mathbf{k}}}$ such that $\alpha\varphi = \psi$. Let ρ be the canonical action of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ on $\mathbb{P}^2_{\overline{\mathbf{k}}}$. Then the action $\varphi\rho\varphi^{-1}$ on $X_{\overline{\mathbf{k}}}$ corresponds to the $\mathbf{k}$-structure X. It follows that the action of $\psi\rho\psi^{-1} = \alpha(\varphi\rho\varphi^{-1})\alpha^{-1}$ on $\mathbb{P}^1_{\overline{\mathbf{k}}} \times \mathbb{P}^1_{\overline{\mathbf{k}}}$ corresponds to a $\mathbf{k}$-structure isomorphic to X. For any $g \in \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$, we have

$$\psi\rho_g\psi^{-1} \colon ([u_0 : u_1], [v_0 : v_1]) \mapsto \begin{cases} ([v_0^g : v_1^g], [u_0^g : u_1^g]), & \text{if } a_1^g = a_2 \\ ([u_0^g : u_1^g], [v_0^g : v_1^g]), & \text{if } a_1^g = a_1. \end{cases}$$

If $L = \mathbf{k}(a_1, a_2)$, which is a quadratic extension of $\mathbf{k}$, then the generator g of $\mathrm{Gal}(L/\mathbf{k})$ exchanges the geometric components of q, so $X \simeq \mathcal{Q}^L$.

(3) The surfaces $\mathcal{Q}^L$ and $\mathcal{Q}^{L'}$ are isomorphic if and only if there exist birational maps $\varphi \colon \mathcal{Q}^L \dashrightarrow \mathbb{P}^2$ and $\varphi' \colon \mathcal{Q}^{L'} \dashrightarrow \mathbb{P}^2$ as in (2) and $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ such that $\varphi^{-1}\alpha\varphi'$ is an isomorphism. This is the case if and only if the base-points of φ^{-1} and $(\varphi')^{-1}$ have the same splitting field. This is equivalent to L and L' being $\mathbf{k}$-isomorphic. □

In order to be complete, we now show an isomorphism from $\mathcal{Q}^L$ to a quadratic surface $\mathcal{R}^L$ in $\mathbb{P}^3$. Later on, we will choose to use or announce claims using coordinates in $\mathcal{Q}^L$ or in $\mathcal{R}^L$ according to practicality.

Lemma 3.3. *Let $L = \mathbf{k}(a_1)$ be a quadratic extensions of $\mathbf{k}$ and let $t^2 + at + \tilde{a} = (t - a_1)(t - a_2) \in \mathbf{k}[t]$ be the minimal polynomial of a_1. The following hold:*

1. *Let $\mathcal{R}^L \subset \mathbb{P}^3_{WXYZ}$ be the quadric surface given by $WZ = X^2 + aXY + \tilde{a}Y^2$. Then*
$$\mathbb{P}^2 \dashrightarrow \mathcal{R}^L, \quad [x : y : z] \mapsto [x^2 + axy + \tilde{a}y^2 : xz : yz : z^2]$$
is birational, and $\mathcal{R}^L$ is isomorphic to $\mathcal{Q}^L$.
2. *The map $\mathcal{Q}^L \longrightarrow \mathcal{R}^L$ given by*
$$([u_0 : u_1], [v_0 : v_1]) \mapsto [u_0 v_0 (a_1 - a_2) : -a_2 u_0 v_1 + a_1 u_1 v_0 : -u_0 v_1 + u_1 v_0 : (a_1 - a_2) u_1 v_1]$$
$$[W : X : Y : Z] \mapsto ([X - a_1 Y : Z], [X - a_2 Y : Z]) = ([W : X - a_2 Y], [W : X - a_1 Y])$$
is an isomorphism over $\mathbf{k}$.
3. *Let $p \in \mathcal{Q}^L$ be a point of degree 2 with splitting field $L' = \mathbf{k}(b_1)$ whose components are not on the same ruling of $\mathcal{Q}^L_L$. Let $t^2 + bt + \tilde{b} = (t - b_1)(t - b_2) \in \mathbf{k}[t]$ be the minimal polynomial of b_1 over $\mathbf{k}$.*
   (a) *Then there is an automorphism of $\mathcal{Q}^L$ (resp. $\mathcal{R}^L$) that sends p respectively onto*
   $$\{([b_1 : 1], [b_1 : 1]), ([b_2 : 1], [b_2 : 1])\}, \quad \{[b_1^2 : b_1 : 0 : 1], [b_2^2 : b_2 : 0 : 1]\}$$
   (b) *The pencil of $(1,1)$-curves in $\mathcal{Q}^L$ through p is given in X^L by the pencil of hyperplanes whose equations are $\lambda(W + bX + \tilde{b}Z) + \mu Y = 0$ for $[\lambda : \mu] \in \mathbb{P}^1$.*

Proof. (1) The given birational map has a single base-point of degree 2, namely $q = \{[a_1 : 1 : 0], [a_2 : 1 : 0]\}$, and it contracts the line $z = 0$. Its image is the quadric surface $\mathcal{R}^L$ given by $WZ = X^2 + aXY + \tilde{a}Y^2$, and the inverse map $\mathcal{R}^L \dashrightarrow \mathbb{P}^2$ is given by the projection from $[1 : 0 : 0 : 0]$. So $\mathcal{R}^L \simeq \mathcal{Q}^L$ by Lemma 3.2(2).

(2) We compose the birational map from (1) and the birational map $\psi\colon \mathbb{P}^2 \dashrightarrow \mathcal{Q}^L$ from the proof of Lemma 3.2(2) whose base-point is $\{[a_1 : 1 : 0], [a_2 : 1 : 0]\}$.

(3a) We see from the description of $\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$ in Lemma 3.5 that we can assume that p is not in the ruling of $\mathcal{Q}^L_L$ passing through $([1 : 1], [1 : 1])$. The birational map $\psi\colon \mathcal{Q}^L \dashrightarrow \mathbb{P}^2$ from the proof of Lemma 3.2(1) sends p onto a point $\psi(p)$ in $\mathbb{P}^2$ that is not collinear with $\{[a_1 : 1 : 0], [a_2 : 1 : 0]\}$. By Lemma 2.6, there exists an element $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ that sends $\psi(p)$ onto $\{[b_1 : 0 : 1], [b_2 : 0 : 1]\}$. Then $\psi^{-1}\alpha\psi \in \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$ and sends p onto $\{([b_1 : 1], [b_1 : 1]), ([b_2 : 1], [b_2 : 1])\}$. We use the isomorphism from (2) to compute its coordinates in $\mathcal{R}^L$.

(3b) The pencil of $(1, 1)$-curves through p is sent by $\psi\colon \mathcal{Q}^L \dashrightarrow \mathbb{P}^2$ onto the pencil of conics through through $[a_1 : 1 : 0], [a_2 : 1 : 0], [b_1 : 0 : 1], [b_2 : 0 : 1]$. It is given by $\lambda(x^2 + axy + bxz + \tilde{a}y^2 + \tilde{b}z^2) + \mu yz$, and corresponds via ψ to the pencil in the claim. □

Remark 3.4. Let $L = \mathbf{k}(a_1)$ be a quadratic extension of $\mathbf{k}$ and let $t^2 + at + \tilde{a} = (t - a_1)(t - a_2) \in \mathbf{k}[t]$ be the minimal polynomial of a_1. Depending on the characteristic of $\mathbf{k}$, we can assume the values of a to be 0 or 1:

- If the characteristic of $\mathbf{k}$ is not 2, then we can assume that $a = 0$, namely via the $\mathbf{k}$-isomorphism $t \mapsto t - a/2$.
- If the characteristic of $\mathbf{k}$ equals 2, then we can assume that $a = 1$. Indeed, as we assume that $\mathbf{k}$ is a perfect field, all elements of $\mathbf{k}$ are squares, and so $a = 0$ does not give an irreducible polynomial over $\mathbf{k}$. The $\mathbf{k}$-isomorphism $t \mapsto t/a$ reduces $a \neq 0$ to $a = 1$.

Lemma 3.5. *Let $L/\mathbf{k}$ be an extension of degree 2 and let g be the generator of* $\mathrm{Gal}(L/\mathbf{k})$. *The group* $\mathrm{Aut}(\mathcal{Q}^L) \simeq \mathrm{Aut}(\mathcal{R}^L)$ *is isomorphic to the* $\mathbf{k}$*-structure on* $\mathrm{Aut}(\mathbb{P}^1_L \times \mathbb{P}^1_L) \simeq \mathrm{Aut}(\mathbb{P}^1_L)^2 \rtimes \langle (u, v) \overset{\tau}{\mapsto} (v, u)\rangle$ *given by the* $\mathrm{Gal}(L/\mathbf{k})$*-action*

$$(A, B, \tau)^g = (B^g, A^g, \tau),$$

where $A \mapsto A^g$ is the canonical $\mathrm{Gal}(L/\mathbf{k})$*-action on* $\mathrm{Aut}(\mathbb{P}^1_L)$. *Furthermore,*

$$\mathrm{Aut}_{\mathbf{k}}(\mathcal{R}^L) \simeq \mathrm{Aut}_{\mathbf{k}}(Q^L) \simeq \{(A, A^g) \mid A \in \mathrm{PGL}_2(L)\} \rtimes \langle\tau\rangle.$$

Proof. Since $\mathcal{Q}^L$ is the $\mathbf{k}$-structure on $Q^L_L \simeq \mathbb{P}^1_L \times \mathbb{P}^1_L$, the $\mathrm{Gal}(L/\mathbf{k})$-action on the algebraic group

$$\mathrm{Aut}_L(\mathcal{Q}^L) = \mathrm{Aut}(\mathbb{P}^1_L \times \mathbb{P}^1_L) \simeq \mathrm{Aut}(\mathbb{P}^1_L)^2 \rtimes \langle\tau\rangle$$

is a $\mathbf{k}$-structure with fixed points $\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$. The automorphism τ commutes with g, and we have

$$(A, B)^g(q^g, p^g) = (A, B)^g(p, q)^g = ((A, B)(p, q))^g = (Ap, Bq)^g = (B^g q^g, A^g p^g)$$

for any $(A, B) \in \mathrm{Aut}(\mathbb{P}^1_L)^2$ and any $(p, q) \in \mathcal{Q}^L$. It follows that $(A, B)^g = (B^g, A^g)$. The group $\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$ is isomorphic to the subgroup of elements of $\mathrm{Aut}(\mathbb{P}^1_L \times \mathbb{P}^1_L)$ commuting with $\mathrm{Gal}(L/\mathbf{k})$, which yields the remaining claim. □

By the following lemma, whenever we contract a curve onto a point of degree 2 in $\mathcal{Q}^L$ with splitting field L, we can choose the point conveniently.

Lemma 3.6.

(1) Let $p \in \mathcal{Q}^L$ be a point of degree 2 whose geometric components are not on the same ruling of $\mathcal{Q}^L_{\overline{\mathbf{k}}} \simeq \mathbb{P}^1_{\overline{\mathbf{k}}} \times \mathbb{P}^1_{\overline{\mathbf{k}}}$ and whose splitting field is L. Then there exists $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$ such that $\alpha(p) = \{([1:0],[0:1]),\ ([0:1],[1:0])\}$.

(2) Let $r, s \in \mathcal{Q}^L(\mathbf{k})$ be two rational points not contained in the same ruling of $\mathcal{Q}^L_{\overline{\mathbf{k}}}$. Then there exists $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$ such that $\alpha(r) = ([1:0],[1:0])$ and $\alpha(s) = ([0:1],[0:1])$.

Proof. Let g be the generator of $\mathrm{Gal}(L/\mathbf{k})$.

(2) We have $r = ([a:b],[a^g:b^g])$ and $s = ([c:d],[c^g:d^g])$ for some $a,b,c,d \in L$, and $ad - cd \neq 0$ because r and s are not on the same ruling of Q_L. It follows that the map $A\colon [u:v] \mapsto [du - cv : -bu + av]$ is contained in $\mathrm{PGL}_2(L)$. Then $(A, A^g) \in \mathrm{Aut}_{\mathbf{k}}(Q)$ and it sends respectively r and s onto $([1:0],[1:0])$ and $([0:1],[0:1])$.

(1) The point p is of the form $\{([a:b],[c:d]),([c^g:d^g],[a^g:b^g])\}$ for some $a,b,c,d \in L$, and $ad^g - bc^g \neq 0$ because its components are not on the same ruling of $\mathcal{Q}^L_L$. It follows that the map A defined by $[u:v] \mapsto [d^g u - c^g v : -bu + av]$ is contained in $\mathrm{PGL}_2(L)$. Then $(A, A^g) \in \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$ and it sends p onto $\{([1:0],[0:1]),([0:1],[1:0])\}$. □

Lemma 3.7. *Let $p = \{p_1,p_2,p_3\}$ and $q = \{q_1,q_2,q_3\}$ be points in $\mathcal{Q}^L$ of degree 3 such that for any $h \in \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ there exists $\sigma \in \mathrm{Sym}_3$ such that $p_i^h = p_{\sigma(i)}$ and $q_i^h = q_{\sigma(i)}$. Suppose that the geometric components of p (resp. of q) are in pairwise distinct rulings of $\mathcal{Q}^L_L$. Then there exists $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$ such that $\alpha(p_i) = q_i$ for $i = 1,2,3$.*

Proof. Let g be the generator of $\mathrm{Gal}(L/\mathbf{k})$. Since p and q are of degree 3, we have $p_i^g = p_i$ and $q_i^g = q_i$ for $i = 1,2,3$, and therefore $p_i = (a_i, a_i^g)$ and $q_i = (b_i, b_i^g)$, $a_i, b_i \in \overline{\mathbf{k}}$, for $i = 1,2,3$. By hypothesis, for any $h \in \mathrm{Gal}(\overline{\mathbf{k}}/L)$ there exists $\sigma \in \mathrm{Sym}_3$ such that $(a_i^h, a_i^{gh}) = p_i^h = q_{\sigma(i)} = (b_{\sigma(i)}, b_{\sigma(i)}^g)$. We apply Remark 2.7 to the $\mathrm{Gal}(\bar{L}/L)$-invariant sets $\{a_1,a_2,a_3\}$ and $\{b_1,b_2,b_3\}$ in $\mathbb{P}^1_L$ and to the $\mathrm{Gal}(\bar{L}/L)$-invariant sets $\{a_1^g,a_2^g,a_3^g\}$ and $\{b_1^g,b_2^g,b_3^g\}$ in $\mathbb{P}^1_L$. There exist $A, B \in \mathrm{PGL}_2(L)$ such that $Aa_i = b_i$ and $Ba_i^g = b_i^g$ for $i = 1,2,3$. Then $A^g a_i^g = (Aa_i)^g = b_i^g = Ba_i^g$ for $i = 1,2,3$, and therefore $B = A^g$. It follows that $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$. □

4. Del Pezzo surfaces of degree 6

In this section, we classify the rational del Pezzo surfaces of degree 6 over a perfect field $\mathbf{k}$ and describe their automorphism groups.

4.1. **Options for rational del Pezzo surfaces of degree 6.** Let X be a rational del Pezzo surface of degree 6. Then $X_{\overline{\mathbf{k}}}$ is the blow up of three points in $\mathbb{P}^2_{\overline{\mathbf{k}}}$, its (-1)-curves are the three exceptional divisors and strict transforms of the lines passing through two of the three points, and they form a hexagon. The hexagon of $X_{\overline{\mathbf{k}}}$ is $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-invariant. The Galois group $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ acts on the hexagon by symmetries, so we have a homomorphism of groups

$$\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}) \xrightarrow{\rho} \mathrm{Sym}_3 \times \mathbb{Z}/2 \subseteq \mathrm{Aut}(\mathrm{NS}(X_{\overline{\mathbf{k}}})).$$

By *hexagon of* X we mean the hexagon of $X_{\overline{\mathbf{k}}}$ endowed with it canonical $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action. The options for the non-trivial action of $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))$ on the hexagon of X are visualised in Figure 1.

The groups $\mathrm{Aut}(X)$ and $\mathrm{Aut}_{\mathbf{k}}(X)$ act by symmetries on the hexagon of $X_{\overline{\mathbf{k}}}$ and X, respectively, which induces homomorphisms

$$\mathrm{Aut}(X) \longrightarrow \mathrm{Sym}_3 \times \mathbb{Z}/2, \qquad \mathrm{Aut}_{\mathbf{k}}(X) \xrightarrow{\hat{\rho}} \mathrm{Sym}_3 \times \mathbb{Z}/2.$$

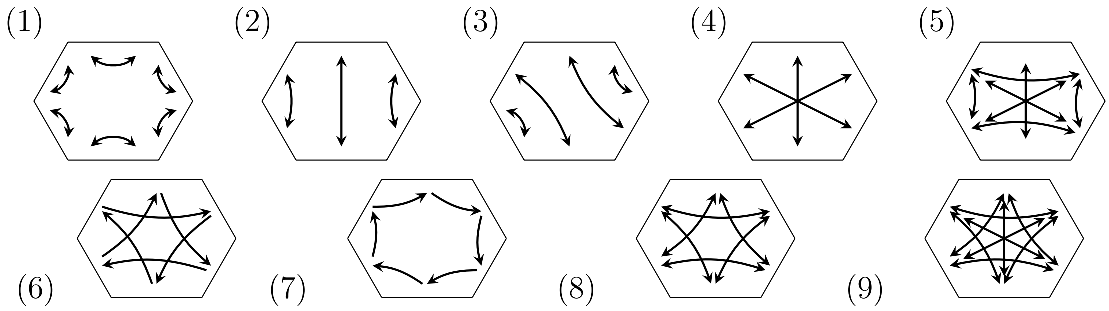


FIGURE 1. The $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-actions on the hexagon of a rational del Pezzo surface of degree 6.

We now go through the cases in Figure 1. We will see that (1), (6), and (8) admit a birational morphism to $\mathbb{P}^2$ and that (2), (3), (4), and (5) admit a birational morphism to $\mathcal{Q}^L$ or $\mathbb{F}_0$.

4.2. **The del Pezzo surfaces in Figures 1(1), 1(6), and 1(8).** The following statement is classical over algebraically closed fields and is proven analogously over a perfect field $\mathbf{k}$.

Lemma 4.1. *Let X be a del Pezzo surface of degree 6 such that $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})) = \{1\}$ as indicated in Figure 1(1)*

(1) Then X is rational and isomorphic to
$$\{([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}^2_{\mathbf{k}} \times \mathbb{P}^2_{\mathbf{k}} \mid x_0y_0 = x_1y_1 = x_2y_2\}.$$
(2) The action of $\mathrm{Aut}_{\mathbf{k}}(X)$ on the hexagon of X induces the split exact sequences
$$1 \to T_2 \to \mathrm{Aut}(X) \to \mathrm{Sym}_3 \times \mathbb{Z}/2 \to 1, \quad 1 \to T_2(\mathbf{k}) \to \mathrm{Aut}_{\mathbf{k}}(X) \xrightarrow{\hat{\rho}} \mathrm{Sym}_3 \times \mathbb{Z}/2 \to 1$$
where T_2 is a 2-dimensional split torus, $\mathbb{Z}/2$ is generated by the image of
$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([y_0 : y_1 : y_2], [x_0 : x_1 : x_2])$$
and Sym_3 is generated by the image of
$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([x_1 : x_0 : x_2], [y_1 : y_0 : y_2])$$
$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto ([x_0 : x_2 : x_1], [y_0 : y_2 : y_1]).$$
(3) $X \longrightarrow *$ is a $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space.

Proof. Contracting three disjoint curves in the hexagon of X yields a birational morphism onto a del Pezzo surface Z of degree 9, and since the images of the three contracted curves are rational points, we have $Z \simeq \mathbb{P}^2$. Choosing the three points to be the coordinate points yields (1). Any element of $\ker(\hat{\rho})$ is conjugate via the contraction to an element of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ fixing the coordinate points and vice-versa, so $\ker(\hat{\rho}) \simeq T_2(\mathbf{k})$. The generators given in (2) can be verified with straightforward calculations. It follows that $\mathrm{Aut}_{\mathbf{k}}(X)$ acts transitively on the sides of the hexagon, hence X is an $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space. □

Over $\overline{\mathbf{k}}$, all rational del Pezzo surfaces of degree 6 are isomorphic. Therefore, by Lemma 4.1, for any del Pezzo surface X of degree 6, we have $\mathrm{rk}\,\mathrm{NS}(X_{\overline{\mathbf{k}}})^{\mathrm{Aut}_{\overline{\mathbf{k}}}(X)} = 1$ and hence X is an $\mathrm{Aut}(X)$-Mori fibre space. Moreover, $\mathrm{Aut}(X)$ is a $\mathbf{k}$-structure on $(\overline{\mathbf{k}}^*)^2 \rtimes (\mathrm{Sym}_3 \times \mathbb{Z}/2)$. We will however encounter two rational del Pezzo surfaces of degree 6 that are not $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre spaces, see Lemma 4.11 and Lemma 4.9.

Lemma 4.2. *Let X be a rational del Pezzo surface of degree 6 such that $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})) = \mathbb{Z}/3$ as indicated in Figure 1(6)*

(1) There exists a point $p = \{p_1, p_2, p_3\}$ in $\mathbb{P}^2$ of degree 3 with splitting field L such that $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathbb{Z}/3$ and such that X is isomorphic to the blow-up of $\mathbb{P}^2$ in p.

(2) X is isomorphic to the graph of a quadratic involution $\varphi_p \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ with base-point p, and any two such surfaces are isomorphic if and only if the corresponding field extensions are $\mathbf{k}$-isomorphic.

(3) The action of $\mathrm{Aut}_{\mathbf{k}}(X)$ on the hexagon of X induces a split exact sequence

$$1 \to \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3) \to \mathrm{Aut}_{\mathbf{k}}(X) \xrightarrow{\hat{\rho}} \mathbb{Z}/6 = \langle \hat{\rho}(\alpha), \hat{\rho}(\beta) \rangle \to 1$$

where α is the lift of an element of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, \{p_1, p_2, p_3\})$ of order 3 and β is the lift of φ_p.

(4) $X \longrightarrow *$ is an $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space.

Proof. (1) The hexagon of X is the union of two curves C_1 and C_2, each of whose three geometric components are disjoint. For $i = 1, 2$, the contraction of C_i yields a birational morphism $\pi_i \colon X \to \mathbb{P}^2$ which contracts the curve onto a point of degree 3. By Lemma 2.6 we can assume it is the same point for $i = 1, 2$, which we call $p = \{p_1, p_2, p_3\}$. It remains to see that $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathbb{Z}/3$, where L is any splitting field of p. Since $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})) \simeq \mathbb{Z}/3$, the action of $\mathrm{Gal}(L/\mathbf{k})$ on $\{p_1, p_2, p_3\}$ induces an exact sequence $1 \longrightarrow H \longrightarrow \mathrm{Gal}(L/\mathbf{k}) \longrightarrow \mathbb{Z}/3 \longrightarrow 1$. The field $L' := \{a \in L \mid h(a) = a \ \forall\ h \in H\}$ is an intermediate field between L and $\mathbf{k}$, over which p_1, p_2, p_3 are rational. The minimality of L implies that $L' = L$ and hence $H = \{1\}$ [27, Corollary 2.10].

(2) The fact that any two such surfaces X are isomorphic if and only if the respective field extensions are $\mathbf{k}$-isomorphic follows from Remark 2.6. The map $\varphi_p := \pi_2\pi_1^{-1} \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ is of degree 2 and p is the base-point of φ_p and φ_p^{-1}. By Lemma 2.6 we can assume that φ_p has a rational fixed point r and that it contracts the line through p_i, p_j onto p_k, where $\{i, j, k\} = \{1, 2, 3\}$. These conditions imply that φ_p is an involution, and by construction of φ_p, the surface X is isomorphic to the graph of φ_p.

(3) The kernel $\ker(\hat{\rho})$ is conjugate via π_1 to the subgroup of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ fixing p_1, p_2, p_3. The only non-trivial elements of $\mathrm{Sym}_3 \times \mathbb{Z}/2$ commuting with $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))$ are rotations, so $\hat{\rho}(\mathrm{Aut}_{\mathbf{k}}(X)) \subseteq \mathbb{Z}/6$. The involution $\varphi_p \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ lifts to an automorphism β inducing a rotation of order 2. If $\langle \sigma \rangle = \mathbb{Z}/3$, there exists $\tilde{\alpha} \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ such that $\tilde{\alpha}(p_i) = p_{\sigma(i)}$, $i = 1, 2, 3$, and $\tilde{\alpha}(r) = r$, where r is the fixed point of φ_p, see Lemma 2.6. Then $\tilde{\alpha}^3$ and $\tilde{\alpha}\varphi_p\tilde{\alpha}^{-1}\varphi_p$ are linear and fix r, p_1, p_2, p_3, and hence $\tilde{\alpha}$ is of order 3 and $\tilde{\alpha}$ and φ_p commute. The lift α of $\tilde{\alpha}$ is an automorphism commuting with β and inducing a rotation of order 3.

(4) Since $\mathrm{Aut}_{\mathbf{k}}(X)$ contains an element inducing a rotation of order 6 on the hexagon, we have $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 1$. □

Lemma 4.3. *Let X be a rational del Pezzo surface of degree 6 such that $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})) = \mathrm{Sym}_3$ as indicated in Figure 1(8)*

(1) There exists a point $p = \{p_1, p_2, p_3\}$ in $\mathbb{P}^2$ of degree 3 with splitting field L such that $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathrm{Sym}_3$ and such that X is isomorphic to the blow-up of $\mathbb{P}^2$ in p.

(2) X is isomorphic to the graph of a quadratic involution $\varphi_p \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ with base-point p, and any two such surfaces are isomorphic if and only if the corresponding field extensions are $\mathbf{k}$-isomorphic.

(3) The action of $\mathrm{Aut}_{\mathbf{k}}(X)$ on the hexagon of X induces a split exact sequence

$$1 \to \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3) \to \mathrm{Aut}_{\mathbf{k}}(X) \xrightarrow{\hat{\rho}} \mathbb{Z}/2 = \langle \hat{\rho}(\alpha) \rangle \to 1$$

where α is the lift of φ_p onto X.

(4) *$X \longrightarrow *$ is an $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space.*

Proof. (1) and (2) are proven analogously to Lemma 4.2(1) and 4.2(2).

(3) The kernel of $\hat{\rho}$ is conjugate to $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3)$ via the birational morphism $X \longrightarrow \mathbb{P}^2$ that contracts one curve in the hexagon of X onto p. Any element of $\mathrm{Aut}_{\mathbf{k}}(X)$ induces a symmetry of the hexagon that commutes with the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action on the hexagon, hence $\hat{\rho}(\mathrm{Aut}_{\mathbf{k}}(X))$ is contained in the factor $\mathbb{Z}/2$ generated by a rotation of order 2. The quadratic involution φ_p lifts to an automorphism α of X and $\hat{\rho}(\alpha)$ is a rotation of order 2.

(4) Since $\hat{\rho}(\alpha)$ exchanges the two curves in the hexagon, we have $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 1$. □

Example 4.4. A del Pezzo surface as in Lemma 4.2 exists: let $|\mathbf{k}| = 2$ and $L/\mathbf{k}$ be the splitting field of $p(X) = X^3 + X + 1$, *i.e.* $|L| = 8$. Then $\sigma\colon a \mapsto a^2$ generates $\mathrm{Gal}(L/\mathbf{k})$ [27, Theorem 6.5]. If ζ a root of P, then $\sigma(\zeta^4) = \zeta$ and hence the point $\{[1 : \zeta : \zeta^4], [1 : \zeta^2 : \zeta], [1 : \zeta^4 : \zeta^2]\}$ is of degree 3, its components are not collinear and they are cyclically permuted by σ.

Example 4.5. A del Pezzo surface as in Lemma 4.3 exists: let $\mathbf{k} = \mathbb{Q}$, $\zeta := 2^{\frac{1}{3}}$ and $\omega = e^{\frac{2\pi i}{3}}$. Then $L := \mathbb{Q}(\zeta, \omega)$ is a Galois extension of $\mathbb{Q}$ of degree 6 and $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathrm{Sym}_3$ is the group of $\mathbf{k}$-isomorphisms of L sending (ζ, ω) respectively to (ζ, ω), $(\omega\zeta, \omega)$, (ζ, ω^2), $(\omega\zeta, \omega^2)$, $(\omega^2\zeta, \omega)$, $(\omega^2\zeta, \omega^2)$ [27, Example 2.21]. The point $\{[\zeta : \zeta^2 : 1], [\omega\zeta : \omega^2\zeta^2 : 1], [\omega^2\zeta : \omega\zeta^2 : 1]\}$ is of degree 3, its components are not collinear and any non-trivial element of $\mathrm{Gal}(L/\mathbf{k})$ permutes them non-trivially.

A del Pezzo surfaces as in Lemma 4.3 cannot exist over a finite field, because Galois groups of finite extensions of finite fields are always cyclic.

4.3. **The del Pezzo surface in Figures 1(7) and 1(9).** Recall that the two del Pezzo surfaces of degree 6 in Lemma 4.2 and Lemma 4.3 are the blow-up of a point $p \in \mathbb{P}^2$ of degree 3.

Lemma 4.6. *Let X be a rational del Pezzo surface with $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})) = \mathbb{Z}/6$ as in Figure 1(7). Then $X \longrightarrow *$ is a Mori fibre space and*

(1) *there exists a quadratic extension $L/\mathbf{k}$ such that X_L is isomorphic to the del Pezzo surface of degree 6 from Lemma 4.2 (see Figure 1(5)), which is the blow-up $\pi\colon X_L \longrightarrow \mathbb{P}^2_L$ of a point $p = \{p_1, p_2, p_3\}$ of degree 3 with splitting field F such that $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathbb{Z}/3$.*

(2) *$\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}$ acts rationally on $\mathbb{P}^2_L$; it is not defined at p, sends a general line onto a conic through p and acts on $\mathrm{Aut}_L(\mathbb{P}^2, \{p_1, p_2, p_3\})$ by conjugation.*

(3) *Any two such surfaces are isomorphic if and only if the corresponding field extensions of degree two and three are $\mathbf{k}$-isomorphic.*

(4) *The action of $\mathrm{Aut}_{\mathbf{k}}(X)$ on the hexagon of X induces a split exact sequence*

$$1 \longrightarrow \mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}} \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathbb{Z}/6 = \langle \hat{\rho}(\alpha), \hat{\rho}(\pi^{-1}\varphi_p\pi) \rangle \longrightarrow 1$$

where α is the lift of an element in $\mathrm{Aut}_L(\mathbb{P}^2, \{p_1, p_2, p_3\})^{\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}}$ of order 3 and $\varphi_p \in \mathrm{Bir}_L(\mathbb{P}^2)$ a quadratic involution with base-point p.

Proof. All (-1)-curves of $X_{\overline{\mathbf{k}}}$ are in the same $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbit and hence $X \longrightarrow *$ is a Mori fibre space.

(1) Since X is rational, it contains a rational point $r \in X(\mathbf{k})$, see Proposition 2.9, which is in particular not contained in the hexagon of X. Let $\eta_1\colon Y \longrightarrow X$ be its blow-up

and E_r its exceptional divisor. Then $Y_{\overline{\mathbf{k}}}$ contains an orbit of three (-1)-curves C_1, C_2, C_3 intersecting E_r, each intersecting two opposite sides of the hexagon. The contraction of $C := C_1 \cup C_2 \cup C_3$ yields a birational morphism $\eta_2 \colon Y \longrightarrow Z$ onto a rational del Pezzo surface of degree 8. The birational map $\eta_2\eta_1^{-1}$ conjugates the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action on Z to an action that exchanges the fibrations of $Z_{\overline{\mathbf{k}}}$ and hence $Z \simeq \mathcal{Q}^L$ for some quadratic extension $L/\mathbf{k}$, by Lemma 3.2(1). Figure 2 shows the action of $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))$ on the image by $\eta_2\eta_1^{-1}$ of the hexagon of X. Then $\eta_2\eta_1^{-1}$ conjugates the $\mathrm{Gal}(\overline{\mathbf{k}}/L)$-action on $\mathcal{Q}^L_L$ to an action on

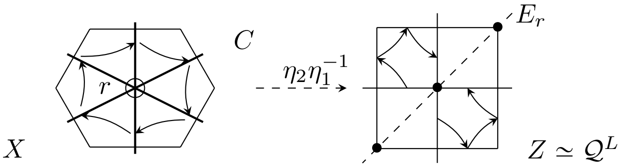


FIGURE 2. The $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action on $Z_{\overline{\mathbf{k}}} \simeq \mathcal{Q}^L_{\overline{\mathbf{k}}}$.

the hexagon with $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/L)) = \mathbb{Z}/3$. Lemma 4.2 implies (1).

(3) By Lemma 3.7, $\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$ acts transitively on the set of points of degree 3 in $\mathcal{Q}^L$ with $\mathbf{k}$-isomorphic splitting fields and whose geometric components are in general position. This yields the claim.

(2) Write $\mathrm{Gal}(L/\mathbf{k}) = \langle g \rangle$. Then g exchanges opposite edges of the hexagon and thus $\rho_g := \pi g \pi^{-1}$ acts rationally on $\mathbb{P}^2$; it is not defined at p, contracts the lines through any two of p_1, p_2, p_3 onto the third of these three and it sends a general line onto a conic through p. It follows that for $\beta \in \mathrm{Aut}_L(\mathbb{P}^2, \{p_1, p_2, p_3\})$ the map $\rho_g \beta \rho_g$ is contained in $\mathrm{Aut}_L(\mathbb{P}^2)$ and preserves $\{p_1, p_2, p_3\}$.

(4) The automorphisms of X are the automorphisms of $X_{\overline{\mathbf{k}}}$ commuting with the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action, hence $\hat{\rho}(\mathrm{Aut}_{\mathbf{k}}(X)) \subseteq \mathbb{Z}/6$. Since X is rational, $\mathrm{Gal}(L/\mathbf{k})$ has a fixed point $r \in X(\mathbf{k})$. Let $\varphi_p \in \mathrm{Bir}_L(\mathbb{P}^2)$ be the quadratic involution from Lemma 4.2(3) such that $\Phi_p := \pi^{-1}\varphi_p\pi \in \mathrm{Aut}_L(X)$ induces a rotation of order 2 on the hexagon of X_L. By Lemma 2.6, we can assume that φ_p fixes $\pi(r) \in \mathbb{P}^2(L)$. Then $\Phi_p g \Phi_p g \in \mathrm{Aut}_L(X)$, preserves the edges of the hexagon and fixes r. It therefore descends to an element of $\mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)$ fixing r and is hence equal to the identity. It follows that $\Phi_p \in \mathrm{Aut}_{\mathbf{k}}(X)$. By Lemma 4.2(3), there is an element of $\tilde{\alpha} \in \mathrm{Aut}_L(\mathbb{P}^2, \{p_1, p_2, p_3\})$ of order 3 inducing a rotation of order 3 on the hexagon of X_L, and again we can assume that it fixes $\pi(r) \in \mathbb{P}^2(L)$. We argue as above that $\alpha := \pi^{-1}\tilde{\alpha}\pi \in \mathrm{Aut}_{\mathbf{k}}(X)$, and it follows that the sequence is split. Finally, any element of $\ker(\hat{\rho})$ preserves each edge of the hexagon and is therefore conjugate by π to an element of $\mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)$ commuting with ρ_g, and any element of $\mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\rho_g}$ lifts to an element of $\ker(\hat{\rho})$. □

Lemma 4.7. *Let X be a rational del Pezzo surface with $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})) = \mathrm{Sym}_3 \times \mathbb{Z}/2$ as in Figure 1(9). Then $X \longrightarrow *$ is a Mori fibre space and*

1. *there exists a quadratic extension $L/\mathbf{k}$ such that X_L is isomorphic to the del Pezzo surface of degree 6 from Lemma 4.3 (see Figure 1(7)), which is the blow-up $\pi \colon X_L \longrightarrow \mathbb{P}^2_L$ of a point $p = \{p_1, p_2, p_3\}$ of degree 3 with splitting field F such that $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3$.*
2. *$\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}$ acts rationally on $\mathbb{P}^2$; it is not defined at p, sends a general line onto a conic through p and acts on $\mathrm{Aut}_L(\mathbb{P}^2, \{p_1, p_2, p_3\})$ by conjugation.*
3. *Any two such surfaces are isomorphic if and only if the corresponding field extensions of degree two and six are $\mathbf{k}$-isomorphic.*

(4) The action of $\mathrm{Aut}_{\mathbf{k}}(X)$ *on the hexagon of* X *induces a split exact sequence*

$$1 \longrightarrow \mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}} \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathbb{Z}/2 = \langle \hat{\rho}(\pi^{-1}\varphi_p\pi)\rangle \longrightarrow 1,$$

where $\varphi_p \in \mathrm{Bir}_L(\mathbb{P}^2)$ *is a quadratic involution with base-point* p.

Proof. This is proven analogously to Lemma 4.6. □

Example 4.8. Rational del Pezzo surfaces of degree 6 over $\mathbf{k}$ as in Lemma 4.6 and Lemma 3.2 exist: in Example 4.4 and Example 4.5, there is a point $p \in \mathbb{P}^2$ of degree 3 with a splitting field $F/\mathbf{k}$ that is Galois over $\mathbf{k}$ such that $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathbb{Z}/3$ or $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3$, and the blow-up $\pi\colon Y \longrightarrow \mathbb{P}^2$ of p is a rational del Pezzo surface of degree 6 as in Figure 1(6) or (8). The point p is also a point of degree 3 in $\mathbb{P}^2_L$ with splitting field FL/L because $\mathrm{Gal}(FL/L) \simeq \mathrm{Gal}(F/\mathbf{k})$ [27, Theorem 5.5].

By Lemma 4.2(2) and Lemma 4.3(2) there exists a quadratic involution $\varphi_p \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ such that $\Phi := \pi^{-1}\varphi_p\pi \in \mathrm{Aut}_{\mathbf{k}}(Y)$ induces a rotation of order 2. By Lemma 2.6, we can assume that φ_p has a rational fixed point $r \in \mathbb{P}^2(\mathbf{k})$. Let g be the generator of $\mathrm{Gal}(L/\mathbf{k})$ and define $\psi_g := \Phi \circ g = g \circ \Phi$. The group $\langle\psi_g\rangle$ acts on Y_L with fixed point $\pi^{-1}(r) \in Y_L(L)$ and it induces a rotation of order 2 on the hexagon of Y_L. It follows that $\mathrm{Gal}(L/\mathbf{k}) \simeq \langle\psi_g\rangle$ defines a $\mathbf{k}$-structure X on Y_L, which is rational by Proposition 2.9. It follows that the group $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ acts on the hexagon of Y_L by $\mathbb{Z}/6$ or by $\mathrm{Sym}_3 \times \mathbb{Z}/2$.

4.4. **The del Pezzo surfaces in Figures 1(3) and 1(4).**

Lemma 4.9. *Let* X *be a del Pezzo surface of degree* 6 *such that* $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))$ *is generated by a reflection as indicated in Figure 1(3). Then* X *is rational and*

1. *there is a quadratic extension* $L/\mathbf{k}$ *and a birational morphism* $\eta\colon X \longrightarrow \mathcal{Q}^L$ *contracting the two* $\mathbf{k}$*-rational curves in the hexagon onto* $p_1 = ([1:0],[1:0])$ *and* $p_2 = ([0:1],[0:1])$.
2. *Any two such surfaces are isomorphic if and only if the respective quadratic extensions are* $\mathbf{k}$*-isomorphic.*
3. *The action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on the hexagon of* X *induces a split exact sequence*
$$1 \to T^L(\mathbf{k}) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \xrightarrow{\hat{\rho}} \langle\hat{\rho}(\alpha)\rangle \times \langle\hat{\rho}(\beta)\rangle \to 1,$$
where $\eta T^L(\mathbf{k})\eta^{-1} \subseteq \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, p_1, p_2)$ *is the subgroup preserving the ruling of* $\mathcal{Q}^L$*, and the automorphisms* $\alpha\colon (u,v) \mapsto (\frac{1}{v}, \frac{1}{u})$ *and* $\beta\colon (u,v) \mapsto (\frac{1}{u}, \frac{1}{v})$.
4. $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 2$ *and* $\eta\,\mathrm{Aut}_{\mathbf{k}}(X)\eta^{-1} = \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, \{p_1, p_2\})$*. In particular,* $X \longrightarrow *$ *is not an* $\mathrm{Aut}_{\mathbf{k}}(X)$*-Mori fibre space.*

Proof. (1) The hexagon of X has exactly two $\mathbf{k}$-rational curves C_1, C_2, which are moreover disjoint. Their contraction yields a birational morphism $\eta\colon X \longrightarrow Z$ onto a del Pezzo surface Z of degree 8 with two rational points. By Proposition 2.9, Z is rational and by Lemma 3.2(1) we have $Z \simeq \mathcal{Q}^L$. We can assume that C_1, C_2 are contracted onto $p_1 = ([1:0],[1:0])$ and $p_2 = ([0:1],[0:1])$ by Lemma 3.6(2).

(2) Any two rational points on $\mathcal{Q}^L$ that are not on the same ruling of $\mathcal{Q}^L_L$ can be sent onto each other by an element of $\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$ by Lemma 3.6(2). It follows that any two del Pezzo surfaces satisfying our hypothesis are isomorphic if and only if they have a birational contraction to isomorphic del Pezzo surfaces $\mathcal{Q}^L$ and $\mathcal{Q}^{L'}$ of degree 8. This is the case if and only if L and L' are $\mathbf{k}$-isomorphic by Lemma 3.2(3).

(3) The kernel of $\hat{\rho}$ is the subgroup of $\mathrm{Aut}_{\mathbf{k}}(X)$ of elements preserving C_1, C_2 and hence its conjugate $\eta\ker(\hat{\rho})\eta^{-1} \subseteq \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, p_1, p_2)$ is the subgroup preserving the rulings of $\mathcal{Q}^L$. The only non-trivial automorphisms of $X_{\overline{\mathbf{k}}}$ commuting with the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action induce

a rotation of order 2 or a reflection that preserves $C_1 \cup C_2$. Let $L/\mathbf{k}$ be an extension of degree 2 such that $\mathcal{Q}_L^L \simeq \mathbb{P}_L^1 \times \mathbb{P}_L^1$. The involution $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$ exchanges p_1, p_2 and the rulings of $\mathcal{Q}_L^L$, it thus lifts to an automorphism of X inducing a reflection. The involution $\beta \in \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$ exchanges p_1, p_2 and preserves the rulings of $\mathcal{Q}_L^L$, it thus lifts to an involution of X inducing a rotation of order 2 on the hexagon. The involutions $\alpha, \beta \in \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$ commute, hence their lifts commute, which yields the splitness of the sequence.

(4) It follows from (3) that any automorphism of X preserves $C_1 \cup C_2$, and since $\eta^{-1}\alpha\eta \in \mathrm{Aut}_{\mathbf{k}}(X)$ exchanges C_1, C_2, we have $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 2$. □

The $\mathbb{R}$-version of Lemma 4.9(3) in [30, Proposition 3.4] states that the kernel is $\mathrm{SO}(\mathbb{R})$, but it should be $T_Q(\mathbb{R}) \simeq \mathrm{SO}(\mathbb{R}) \times \mathbb{R}_{>0}$.

Lemma 4.10. *Let X be a rational del Pezzo surface of degree 6 such that $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))$ is generated by a rotation of order 2 as indicated in Figure 1(4). Then there exists a quadratic extension $L = \mathbf{k}(a_1)$ of $\mathbf{k}$ such that*

(1) X is isomorphic to the blow-up of $\mathbb{F}_0$ in the point $\{[a_1 : 1; a_1 : 1], [a_2 : 1; a_2 : 1]\}$ of degree 2 and

$$X \simeq \{([u_0 : u_1], [v_0 : v_1], [w_0 : w_1]) \in (\mathbb{P}^1)^3 \mid w_0\tilde{a}(u_0v_0 + au_1v_0 + \tilde{a}u_1v_1) = w_1(u_0v_1 - x_1v_0)\}$$

where $t^2 + at + \tilde{a} = (t - a_1)(t - a_2) \in \mathbf{k}[t]$ is the minimal polynomial of a_1 over $\mathbf{k}$.

(2) Any two such surfaces are isomorphic if and only if the respective quadratic extensions are $\mathbf{k}$-isomorphic.

(3) The action of $\mathrm{Aut}_{\mathbf{k}}(X)$ on the hexagon induces an exact sequence,

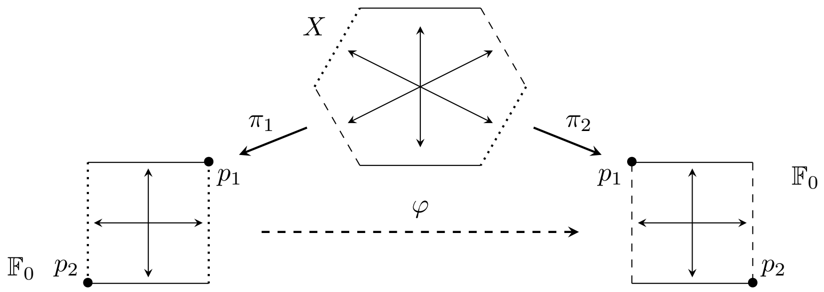
$$1 \to \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, p_1, p_2)^2 \to \mathrm{Aut}_{\mathbf{k}}(X) \xrightarrow{\hat{\rho}} \mathrm{Sym}_3 \times \mathbb{Z}/2 \to 1,$$

which is split if $\mathrm{char}(\mathbf{k}) \neq 2$, $\mathbb{Z}/2 = \langle \hat{\rho}(\tilde{\alpha}) \rangle$ and $\mathrm{Sym}_3 = \langle \hat{\rho}(\tilde{\beta}), \hat{\rho}(\tilde{\varphi}) \rangle$, where $\tilde{\alpha}, \tilde{\beta}, \tilde{\varphi}$ are the lifts of the involutions of $\mathbb{F}_0$

$$\begin{aligned} \alpha\colon [y_0 : y_1; z_0 : z_1] &\mapsto [y_0 + ay_1 : -y_1; z_0 + az_1 : -z_1], \\ \beta\colon [y_0 : y_1; z_0 : z_1] &\mapsto [z_0 : z_1; y_0 : y_1], \\ \psi\colon [y_0 : y_1; z_0 : z_1] &\dashrightarrow [y_0 + ay_1 : -y_1; \tilde{a}(y_1z_0 - y_0z_1) : y_0z_0 + ay_0z_1 + \tilde{a}y_1z_1]. \end{aligned}$$

(4) $X \longrightarrow *$ is an $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space.

Proof. (1) Let C_1, C_2, C_3 be the curves in the hexagon of X. By Lemma 3.2(1), for $i = 1, 2, 3$, there is a birational morphism $\pi_i\colon X \to \mathbb{F}_0$ contracting C_i onto a point of degree 2. Let $L/\mathbf{k}$ be a quadratic extension such that $\mathrm{Gal}(L/\mathbf{k})$ acts by the rotation of order 2. Then $\mathrm{Gal}(\overline{\mathbf{k}}/L)$ preserves each C_i, hence L is the splitting field of each C_i. So, L is also the splitting field of each $\pi_i(C_i)$. Let $L = \mathbf{k}(a_1)$ for some $a_1 \in L$. For $i = 1, 2, 3$ we write $\pi_i(C_i) = \{[b_{i1} : 1; b_{i2} : 1], [b_{i3} : 1; b_{i4} : 1]\}$ for some $b_{i1}, \ldots, b_{i4} \in L$. Since the two components of $\pi_i(C_i)$ are not contained in the same fibre of $\mathbb{F}_0$, Remark 2.7 implies that there is $A_i \in \mathrm{PGL}_2(\mathbf{k})$ that sends $[b_{i1} : 1], [b_{i3} : 1]$ onto $[a_1 : 1], [a_2 : 1]$. Similarly, there is $B_i \in \mathrm{PGL}_2(\mathbf{k})$ that sends $[b_{i2} : 1], [b_{i4} : 1]$ onto $[a_1 : 1], [a_2 : 1]$. Up to changing the rulings on $\mathbb{F}_0$, we can assume that $\varphi := \pi_2\pi_1^{-1}\colon \mathbb{F}_0 \dashrightarrow \mathbb{F}_0$ preserves the ruling given by the first projection, as indicated in the following commutative diagram.



Up to an isomorphism of the first factor, we can assume that φ induces the identity map on $\mathbb{P}^1$. It then sends a general fibre f of the second projection onto a curve of bidegree $(1,1)$ passing through q, which is given by $\lambda(y_0z_1 - y_1z_0) + \mu(y_0z_0 + ay_1z_0 + \tilde{a}y_1z_1) = 0$ for some $[\lambda : \mu] \in \mathbb{P}^1$. So, up to left-composition by an automorphism of the second factor, φ is the involution given by

$$\varphi\colon [y_0 : y_1; z_0 : z_1] \dashrightarrow [y_0 : y_1; \tilde{a}(y_0z_1 - y_1z_0) : y_0z_0 + ay_1z_0 + \tilde{a}y_1z_1].$$

By construction of φ, X is isomorphic to its graph inside $(\mathbb{P}^1)^4$. The projection forgetting the third factor induces the isomorphism in (1).

(2) As indicated in (1), any two points of degree 2 in $\mathbb{F}_0$ whose geometric components are not in the same ruling can be sent onto each other by an element of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_0)$. It follows that two del Pezzo surfaces X and X' satisfying the hypothesis of our lemma are isomorphic if and only if there are contractions $X \longrightarrow \mathbb{F}_0$ and $X' \longrightarrow \mathbb{F}_0$ that contract a curve in each hexagon onto points with $\mathbf{k}$-isomorphic splitting fields. This is equivalent to contracted curves having $\mathbf{k}$-isomorphic splitting fields.

(3) The group $\pi_1 \ker(\hat{\rho})\pi_1^{-1}$ is the subgroup of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_0)$ fixing $[a_i : 1; a_i : 1]$ for $i = 1,2$ and preserving the fibration given by the first projection, hence $\pi_1 \ker(\hat{\rho})\pi_1^{-1} \simeq \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, [a_1 : 1], [a_2 : 1])^2$. The involution $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_0)$ (it is not the identity map by Remark 3.4) preserves the fibrations of $\mathbb{F}_0$ and exchanges $[a_1 : 1; a_1 : 1]$ and $[a_2 : 1; a_2 : 1]$. Thus it lifts to an involution $\tilde{\alpha} \in \mathrm{Aut}_{\mathbf{k}}(X)$ inducing a rotation of order 2 on the hexagon. The involution $\beta \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_0)$ exchanges the fibrations of $\mathbb{F}_0$ and fixes $[a_i : 1; a_i : 1]$ for $i = 1,2$, thus lifts to an involution $\tilde{\beta} \in \mathrm{Aut}_{\mathbf{k}}(X)$ inducing the reflection at the axis through C_1. We check that $\psi := \varphi \circ \alpha$. Since φ induces the reflection on the hexagon that exchanges the components of C_3, ψ induces the reflection preserving each component of C_3. It follows that the sequence is exact. If $\mathrm{char}(\mathbf{k}) \neq 2$, we have $a = 0$, and then ψ is an involution, α commutes with β and ψ, and $\beta \circ \psi$ has order 3. It follows that the sequence is split.

(4) Since $\mathrm{Aut}_{\mathbf{k}}(X)$ acts transitively on the edges of the hexagon, $X \longrightarrow *$ is an $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space. □

4.5. **The del Pezzo surfaces in Figures 1(2) and 1(5).** Here, we consider the remaining two del Pezzo surfaces of degree 6 from Figure 1. We will see that none of them is a $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space. However, they carry a conic fibration, and we will describe the automorphism group preserving the fibration in this section, which will be used in the Section 5.

Lemma 4.11. *Let X be a rational del Pezzo surface of degree 6 such that $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))$ is generated by a reflection as indicated in Figure 1(2). There exists a quadratic extension $L = \mathbf{k}(a_1)/\mathbf{k}$ such that the following holds:*

(1) *There is a birational morphism* $\eta\colon X \longrightarrow \mathcal{R}^L \simeq \mathcal{Q}^L$ *contracting an irreducible* E *curve onto the point* $\eta(E) = \{[a_1^2 : a_1 : 1 : 0], [a_2^2 : a_2 : 1 : 0]\} = \{p_1, p_2\}$ *of degree* 2.
(2) $X \simeq \{([w : x : y : z], [u : v]) \mid v(w + ax + \tilde{a}z) = uy\} \subset \mathcal{R}^L \times \mathbb{P}^1$
(3) *The action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on the hexagon of* X *induces a split exact sequence*

$$1 \to T^{L,L}(\mathbf{k}) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \xrightarrow{\hat{\rho}} \langle \hat{\rho}(\alpha)\rangle \times \langle \hat{\rho}(\beta)\rangle \to 1$$

where $T^{L,L}(\mathbf{k}) \subset \mathrm{Aut}_{\mathbf{k}}(\mathcal{R}^L, p_1, p_2)$ *is the subgroup preserving the rulings of* $\mathcal{R}^L_L$*, and* $\hat{\rho}(\alpha)$ *is the reflection exchanging the singular fibres and* $\hat{\rho}(\beta)$ *is a rotation of order* 2 *with*

$$\eta\alpha\eta^{-1}\colon [w : x : y : z] \mapsto [w : x + ay : -y : z]$$
$$\eta\beta\eta^{-1}\colon [w : x : y : z] \mapsto [w + a(2x + az + ay) : -(x + az) : -y : z]$$

where $t^2 + at + \tilde{a} = (t - a_1)(t - a_2) \in \mathbf{k}[t]$ *is the minimal polynomial of* a_1 *over* $\mathbf{k}$.
(4) *We have* $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 2$ *and* $\eta\,\mathrm{Aut}_{\mathbf{k}}(X)\eta^{-1} = \mathrm{Aut}_{\mathbf{k}}(\mathcal{R}^L, \{p_1, p_2\})$*. In particular,* $X \longrightarrow *$ *is not an* $\mathrm{Aut}_{\mathbf{k}}(X)$*-Mori fibre space.*

Proof. (1) By Lemma 3.2(1), contracting E yields a birational morphism $\nu\colon X \longrightarrow \mathcal{Q}^L$. The splitting field of the image of E is L, so we can choose $\nu(E) = \{([1:0],[0:1]),([0:1],[1:0])\}$ by Lemma 3.6(1). Changing the model of $\mathcal{Q}^L$ with the isomorphism from Lemma 3.3(2), we get the birational morphism $\eta\colon X \longrightarrow \mathcal{R}^L$ and $\eta(E) = \{[a_1^2 : a_1 : 1 : 0], [a_2^2 : a_2 : 1 : 0]\}$.

(4) Any element of $\mathrm{Aut}_{\mathbf{k}}(X)$ preserves E. It follows that $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 2$ and that $\nu\,\mathrm{Aut}_{\mathbf{k}}(X)\nu^{-1} = \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, \{p_1, p_2\})$.

(3) The conjugate $\nu \ker(\hat{\rho})\nu^{-1} \subseteq \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, ([1:0],[0:1]),([0:1],[1:0]))$ is the subgroup preserving the rulings of $\mathcal{Q}^L_L$. The only non-trivial symmetries in $\mathrm{Sym}_3 \times \mathbb{Z}/2$ commuting with the $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))$-action are the two reflections preserving E and the rotation of order 2. By Remark 3.4, $\eta\alpha\eta^{-1}, \eta\beta\eta^{-1}$ are involutions and they commute. Moreover, they respectively fix and exchange $[a_1^2 : a_1 : 1 : 0], [a_2^2 : a_2 : 1 : 0]$. Their conjugates by the isomorphism $\mathcal{R}^L \longrightarrow \mathcal{Q}^L$ from Lemma 3.3(2) respectively exchange and preserve the rulings of $\mathcal{Q}^L_L$. In particular, they induce the claimed action on the hexagon of X, thus the sequence is split. $\square$

Lemma 4.12. *Let* X *be a rational del Pezzo surface of degree* 6 *such that* $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ *is generated by a reflection and a rotation of order* 2 *as in Figure 1(5). Then there exist quadratic extensions* $L = \mathbf{k}(a_1)$ *and* $L' = \mathbf{k}(b_1)$ *of* $\mathbf{k}$ *that are not* $\mathbf{k}$*-isomorphic, with*

$$t^2 + at + \tilde{a} = (t - a_1)(t - a_2), \quad t^2 + bt + \tilde{b} = (t - b_1)(t - b_2) \in \mathbf{k}[t]$$

the minimal polynomials of a_1, b_1 *such that the following hold:*

(1) $X \simeq \mathcal{S}^{L,L'}$ *and there exists a birational contraction* $\eta\colon X \longrightarrow \mathcal{Q}^L \simeq \mathcal{R}^L$ *contracting an irreducible curve onto the point* $\{p_1, p_2\} = \{[b_1^2 : b_1 : 0 : 1], [b_2^2 : b_2 : 0 : 1]\}$ *of degree* 2.
(2) $X \simeq \{([w : x : y : z], [u : v]) \mid v(w + bx + \tilde{b}z) = uy\} \subset \mathcal{R}^L \times \mathbb{P}^1$
(3) *Two surfaces* $\mathcal{S}^{L,L'}$ *and* $\mathcal{S}^{\tilde{L},\tilde{L}'}$ *are isomorphic if and only if* $\tilde{L}, \tilde{L}'$ *are respectively* $\mathbf{k}$*-isomorphic to* L, L'.
(4) *The action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on the hexagon of* X *induces a split exact sequence*

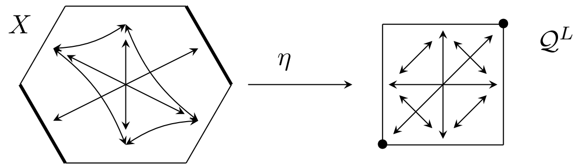
$$1 \longrightarrow T^{L,L'} \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \xrightarrow{\hat{\rho}} \langle \hat{\rho}(\alpha)\rangle \times \langle \hat{\rho}(\beta)\rangle \longrightarrow 1$$

where $T^{L,L'} \subset \mathrm{Aut}_{\mathbf{k}}(\mathcal{R}^L, p_1, p_2)$ is the subgroup preserving the rulings of $\mathcal{R}^L_L$, and $\hat\rho(\alpha)$ is the reflection exchanging the singular fibres and $\hat\rho(\beta)$ is a rotation of order 2, where

$$\eta\alpha\eta^{-1}\colon [w:x:y:z] \mapsto [w:x+ay:-y:z]$$
$$\eta\beta\eta^{-1}\colon [w:x:y:z] \mapsto [w+b(2x+bz+ay):-(x+bz):-y:z]$$

(5) $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 2$ and $\eta\,\mathrm{Aut}_{\mathbf{k}}(X)\eta^{-1} = \mathrm{Aut}_{\mathbf{k}}(\mathcal{R}^L, \{p_1, p_2\})$. In particular, $X \longrightarrow *$ is not an $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space.

Proof. (1) The hexagon of X contains a unique curve E whose geometric components are disjoint. The contraction of E yields a birational morphism $\eta\colon X \longrightarrow Y$ to a del Pezzo surface Y of degree 8, and the figure below shows the induced $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action on the image of the hexagon, so $Y \simeq \mathcal{Q}^L$ for some quadratic extension $L/\mathbf{k}$ by Lemma 3.2(1).



We have $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})) = \{1, r, s, rs\}$, where r is the rotation of order 2 and s is the reflection preserving the components of E. Then s or sr is the image of the generator g of $\mathrm{Gal}(L/\mathbf{k})$. It follows that the splitting field of p is a quadratic extension $L'/\mathbf{k}$ not $\mathbf{k}$-isomorphic to L such that the generator g' of $\mathrm{Gal}(L'/\mathbf{k})$ induces the rotation r on the hexagon. We set $L = \mathbf{k}(a_1)$ and $L' = \mathbf{k}(b_1)$ for some $a_1 \in L$, $b_1 \in L'$. We can choose the form of p according to Lemma 3.3(3a).

(2) follows from (1) and Lemma 3.3(3b).

(3) Consider the birational morphism $\eta'\colon \mathcal{S}^{\tilde L,\tilde L'} \longrightarrow \mathcal{R}^{\tilde L}$ with exceptional curve E'. Suppose that we have $\mathcal{S}^{\tilde L,\tilde L'} \simeq \mathcal{S}^{L,L'}$. Then E and E' are the unique curves in the hexagon with only two components. Thus they are defined over the same splitting field over $\mathbf{k}$, and hence $L' \simeq \tilde L'$ over $\mathbf{k}$. It follows that $\mathcal{R}^L \simeq \mathcal{R}^{\tilde L}$, which implies that $L \simeq \tilde L$ over $\mathbf{k}$ by Lemma 3.2(3).

(4–5) The group $\ker(\hat\rho) \simeq \eta\ker(\hat\rho)\eta^{-1} \subset \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, p_1, p_2)$ is the subgroup preserving the rulings of $\mathcal{Q}^L$. Every element of $\mathrm{Aut}_{\mathbf{k}}(X)$ preserves E because it is the only curve in the hexagon with only two geometric components, so the elements of $\mathrm{Aut}_{\mathbf{k}}(X)$ act by symmetries of order 2, and we have $\eta\,\mathrm{Aut}_{\mathbf{k}}(X)\eta^{-1} = \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, \{p_1, p_2\})$. The only symmetries of the hexagon that commute with $\rho(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))$ are the two reflections preserving E and the rotation of order 2. By Remark 3.4, $\eta\alpha\eta^{-1}, \eta\beta\eta^{-1}$ are involutions and they commute. Moreover, they respectively fix and exchange $[b_1^2 : b_1 : 1 : 0], [b_2^2 : b_2 : 1 : 0]$. We see that the conjugates of $\eta\alpha\eta^{-1}, \eta\beta\eta^{-1}$ by the isomorphism $\mathcal{R}^L \dashrightarrow \mathcal{Q}^L$ from Lemma 3.3(2) respectively exchange and preserve the rulings of $\mathcal{Q}^L_L$. In particular, they induce the claimed action on the hexagon, thus the sequence is split. $\square$

4.6. **The fibration on a rational del Pezzo surface of degree 6 from Figures 1(2) and 1(5).** Let $L/\mathbf{k}$, $L'/\mathbf{k}$ be two extensions of degree 2. We can obtain the Mori fibre space $\pi\colon \mathcal{S}^{L,L'} \longrightarrow \mathbb{P}^1$ from Example 2.5(2) as follows: we first blow up the point p, then contract the line passing through it, which yields a birational map $\mathbb{P}^2 \dashrightarrow \mathcal{Q}^L$. Since p, p' are not collinear, the image of p' in $\mathcal{Q}^L$ is a proper point and blowing it up yields $\mathcal{S}^{L,L'}$. In particular, $\mathcal{S}^{L,L'}$ is one of the del Pezzo surfaces in Figure 1(2) and (5), which are described in Lemma 4.11 and Lemma 4.12.

Remark 4.13.

(1) Let $L = \mathbf{k}(a_1)$ and $L' = \mathbf{k}(b_1)$ be two quadratic extensions of $\mathbf{k}$, not necessarily non-isomorphic over $\mathbf{k}$, and let
$$t^2 + at + \tilde{a} = (t - a_1)(t - a_2), \quad t^2 + bt + \tilde{b} = (t - b_1)(t - b_2) \in \mathbf{k}[t]$$
be the minimal polynomials of a_1 and b_1 over $\mathbf{k}$. Lemma 4.11(2) and Lemma 4.12(2) imply that
$$\mathcal{S}^{L,L'} \simeq \{([w : x : y : z], [u : v]) \in \mathbb{P}^3 \times \mathbb{P}^1 \mid wz = x^2 + axy + \tilde{a}y^2,\ (w + bx + \tilde{b}z)v = uy\}$$
and the fibration $\pi\colon \mathcal{S}^{L,L'} \longrightarrow \mathbb{P}^1$ is given by the projection
$$([w : x : y : z], [u : v]) \mapsto [u : v] = [w + bx + \tilde{b}z : y].$$

(2) The group $\mathrm{Aut}(\mathcal{S}^{L,L'}, \pi)$ preserves a unique irreducible curve E in the hexagon of X that has disjoint geometric components. It induces a morphism
$$\mathrm{Aut}(\mathcal{S}^{L,L'}, \pi) \longrightarrow \mathbb{Z}/2,$$
and we denote by $\mathrm{SO}^{L,L'} \subset \mathrm{Aut}(\mathcal{S}^{L,L'}, \pi)$ its kernel.

(3) Via the contraction $\eta\colon X \longrightarrow \mathcal{Q}^L \simeq \mathcal{R}^L$ of E onto a point $\{p_1, p_2\}$ of degree 2, the group $\mathrm{SO}^{L,L'}$ is conjugate to a subgroup of $T^{L,L'}$, the subgroup of $\mathrm{Aut}(\mathcal{Q}^L, p_1, p_2)$ preserving the rulings of $\mathcal{Q}^L_L$ (see Lemma 4.11(3) and Lemma 4.12(4)).

(4) The image $t, s \in \mathbb{P}^1(L)$ of the singular fibres make up two points of degree 1 if L, L' are $\mathbf{k}$-isomorphic, and one point of degree 2 if L, L' are not $\mathbf{k}$-isomorphic.

Lemma 4.14. *Keep the notation of Remark 4.13 and let g be the generator of $\mathrm{Gal}(L/\mathbf{k})$. Then the action of $\mathrm{Aut}(\mathcal{S}^{L,L'}/\pi)$ on the geometric components of E induces the split exact sequences*
$$1 \to \mathrm{SO}^{L,L'} \longrightarrow \mathrm{Aut}(\mathcal{S}^{L,L'}/\pi) \longrightarrow \mathbb{Z}/2 \to 1$$
$$1 \to \mathrm{SO}^{L,L'}(\mathbf{k}) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(\mathcal{S}^{L,L'}/\pi) \longrightarrow \mathbb{Z}/2 \to 1$$
where $\mathbb{Z}/2$ is generated by the image of the involution
$$([w : x : y : z], [u : v]) \mapsto ([w + b(2x + ay + bz) : -(x + ay + bz) : y : z], [u : v]),$$
and $\mathrm{SO}^{L,L'} \simeq \{(\alpha, \beta) \in T^{L,L'} \mid \alpha\beta = 1\}$, whose $\mathbf{k}$-rational points are given by

(1) either $\mathrm{SO}^{L,L}(\mathbf{k}) \simeq \{\alpha \in L^* \mid \alpha\alpha^g = 1\}$,
(2) or $\mathrm{SO}^{L,L'}(\mathbf{k}) \simeq \mathbf{k}^*$ if L, L' are not $\mathbf{k}$-isomorphic.

Proof. The indicated map is the composition of the two commuting involutions α, β from Lemma 4.11(3) and Lemma 4.12(4). In particular, it is an involution (it is not the identity by Remark 3.4) that induces a reflection on the hexagon exchanging the geometric components of the singular fibres.

Let us compute the image of $\mathrm{SO}^{L,L'}$ in $T^{L,L'}$. Since this means computing the $\overline{\mathbf{k}}$-points of these groups, it suffices to assume that L and L' are $\mathbf{k}$-isomorphic. We consider $\mathcal{Q}^L$ as $\mathbf{k}$-structure on $\mathbb{P}^1_L \times \mathbb{P}^1_L$. By Lemma 3.6(1), we can assume that $p_1 = ([0 : 1], [1 : 0])$, $p_2 = ([1 : 0], [0 : 1])$. Then $\mathrm{SO}^{L,L}$ is conjugate to a subgroup of the group of diagonal maps $\mathrm{Aut}(\mathcal{Q}^L, p_1, p_2)$. In these coordinates, the fibration $\pi\colon \mathcal{S}^{L,L} \longrightarrow \mathbb{P}^1$ is mapped by η to the pencil of curves given by $cu_1v_1 - du_0v_0 = 0$, $[c : d] \in \mathbb{P}^1$. A diagonal element $(\alpha, \beta) \in \mathrm{Aut}(\mathcal{Q}^L, p_1, p_2)$ preserves each fibre if and only if $\alpha\beta = 1$. It follows that $\mathrm{SO}^{L,L} = \{(\alpha, \beta) \in T^{L,L} \mid \alpha\beta = 1\}$.

(1) The $\mathbf{k}$-rational points $\mathrm{SO}^{L,L}(\mathbf{k})$ form the subgroup of elements in $\mathrm{SO}^{L,L'}(\overline{\mathbf{k}})$ that are fixed by the $\mathrm{Gal}(L/\mathbf{k})$-action, see Lemma 3.5. The generator $g \in \mathrm{Gal}(L/\mathbf{k})$ acts by $(\alpha,\beta)^g = (\beta^g,\alpha^g)$, see Lemma 3.5. It follows that $\mathrm{SO}^{L,L} = \{(\alpha,\beta) \in T^{L,L} \mid \alpha\beta = 1\}$.

(2) Suppose that L, L' are not $\mathbf{k}$-isomorphic. Let $K := LL'$. Then $\mathrm{Gal}(K/\mathbf{k}) \simeq \mathrm{Gal}(L/\mathbf{k})\times \mathrm{Gal}(L'/\mathbf{k})$. Lemma 3.3(3a) tells us that we can assume that $p_1 = ([b_1:1],[b_1:1]), p_2 = ([b_2:1],[b_2:1])$. We now compute the form of the elements in $\mathrm{SO}^{L,L'}(K)$: the element

$$\gamma := \left(\begin{pmatrix} b_2 & b_1 \\ 1 & 1\end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ 1 & 1\end{pmatrix}\right) \in \mathrm{PGL}_2(K)\times\mathrm{PGL}_2(K).$$

induces a change of coordinates $\gamma\colon \mathcal{Q}_K^L \longrightarrow \mathbb{Q}_L^K$ sending $([0:1],[1:0]),([1:0],[0:1])$ onto p_1,p_2, respectively. Then $\mathrm{SO}^{L,L'}(K) \subset \mathrm{PGL}_2(K)^2$ is the subgroup of elements of the form

$$\text{(AB)}\quad (A,B) := \gamma\circ(\alpha,\beta)\circ\gamma^{-1} = \left(\begin{pmatrix} b_2\alpha - b_1 & b_1b_2(1-\alpha) \\ \alpha-1 & b_2-\alpha b_1\end{pmatrix}, \begin{pmatrix} b_1\beta - b_2 & b_1b_2(1-\beta) \\ \beta-1 & b_1-b_2\beta\end{pmatrix}\right).$$

The group $\mathrm{SO}^{L,L'}(\mathbf{k})$ is the $\mathrm{Gal}(K/\mathbf{k})$-invariant subgroup of $\mathrm{SO}^{L,L'}(K)$. If g is the generator of $\mathrm{Gal}(L/\mathbf{k})$, and g' is the one of $\mathrm{Gal}(L'/g)$, then

$$(A,B)^g = (B^g, A^g), \quad (A,B)^g = (A^{g'}, B^{g'})$$

It follows that

$$\mathrm{SO}^{L,L'}(\mathbf{k}) = \{(A,B) \in \mathrm{PGL}_2(L')^2 \mid (A,B) \text{ of the form (AB)},\ \alpha\beta^g = 1 = \alpha\beta\}$$

We obtain that $\beta \in \mathbf{k}^*$, and hence that $\mathrm{SO}^{L,L'}(\mathbf{k}) \simeq \mathbf{k}^*$. □

Lemma 4.15. *Keep the notation of Remark 4.13 and let g be the generator of* $\mathrm{Gal}(L/\mathbf{k})$. *Then the action of* $\mathrm{Aut}(\mathcal{S}^{L,L'},\pi)$ *on* $\mathbb{P}^1$ *induces the exact sequences*

$$1 \to \mathrm{Aut}(\mathcal{S}^{L,L'}/\pi) \longrightarrow \mathrm{Aut}(\mathcal{S}^{L,L'},\pi) \longrightarrow \mathrm{Aut}(\mathbb{P}^1,\{t,s\}) \simeq T_1 \rtimes \mathbb{Z}/2 \to 1$$
$$1 \to \mathrm{Aut}_{\mathbf{k}}(\mathcal{S}^{L,L'}/\pi) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(\mathcal{S}^{L,L'},\pi) \longrightarrow D_{\mathbf{k}}^{L,L'} \rtimes \mathbb{Z}/2 \to 1$$

where T_1 is the 1-dimensional split torus, $\mathbb{Z}/2$ is generated by the image of

$$([w:x:y:z],[u:v]) \mapsto ([w+b(2x+ay+bz) : -(x+bz) : -y : z],[u+abv:-v])$$

and $D_{\mathbf{k}}^{L,L'} \subseteq T_1(\mathbf{k})$ is the subgroup

(1) $D_{\mathbf{k}}^{L,L} = \{\delta \in T_1(\mathbf{k}) \mid \delta = \lambda\lambda^g, \lambda \in L^*\}$, *where g is the generator of* $\mathrm{Gal}(L/\mathbf{k})$,

(2) $D_{\mathbf{k}}^{L,L'} \simeq \{\lambda\lambda^{gg'} \in F \mid \lambda \in K, \lambda\lambda^{g'} = 1\}$ *if L and L' are not $\mathbf{k}$-isomorphic, where $\mathbf{k} \subset F \subset LL'$ is the intermediate extension such that* $\mathrm{Gal}(F/\mathbf{k}) \simeq \langle gg'\rangle \subset \mathrm{Gal}(L/\mathbf{k}) \times \mathrm{Gal}(L'/\mathbf{k})$, *where g, g' are the generators of* $\mathrm{Gal}(L/\mathbf{k}), \mathrm{Gal}(L'/\mathbf{k})$, *respectively.*

Proof. The birational contraction $\eta\colon \mathcal{S}^{L,L'} \longrightarrow \mathcal{Q}^L$ induces a rational map $\hat{\pi}\colon \mathcal{Q}^L \dashrightarrow \mathbb{P}^1$ such that $\hat{\pi}\circ\eta = \pi$. We define

$$\mathrm{Aut}(\mathcal{Q}^L,\hat{\pi}) = \{\alpha \in \mathrm{Aut}(\mathcal{Q}^L) \mid \exists f \in \mathrm{Aut}(\mathbb{P}^1) \text{ such that } \hat{\pi}\circ\alpha = f\circ\hat{\pi}\}$$

Then $\mathrm{Aut}(\mathcal{Q}^L,\hat{\pi}) = \eta\,\mathrm{Aut}(\mathcal{S}^{L,L'},\pi)\eta^{-1}$. Let us compute $\mathrm{Aut}_{\overline{\mathbf{k}}}(\mathcal{Q}^L_{\overline{\mathbf{k}}},\hat{\pi})$. For this, we can assume that $p_1 = ([0:1],[1:0])$, $p_2 = ([1:0],[0:1])$ (in the notation of Remark 4.13), and the fibres of $\hat{\pi}$ are of the form $cu_1v_1 - du_0v_0 = 0$, $[c:d] \in \mathbb{P}^1$. It follows that

$$\mathrm{Aut}_{\overline{\mathbf{k}}}(\mathbb{P}^1_{\overline{\mathbf{k}}}\times\mathbb{P}^1_{\overline{\mathbf{k}}}) \supseteq \mathrm{Aut}_{\overline{\mathbf{k}}}(\mathcal{Q}^L_{\overline{\mathbf{k}}},\hat{\pi}) = \left\{(A_\lambda,B_\mu) \mid \lambda,\mu \in \overline{\mathbf{k}}^*\right\} \rtimes \langle\tau\colon (x,y)\mapsto(y,x)\rangle$$

where

$$(I)\ A_\lambda = \begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}, B_\mu = \begin{pmatrix}1 & 0\\ 0 & \mu\end{pmatrix}, \qquad \text{or} \qquad (II)\ A_\lambda = \begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}, B_\mu = \begin{pmatrix}0 & 1\\ \mu & 0\end{pmatrix}.$$

The automorphism (A_λ, B_μ) of type (I) induces the scaling $[c:d] \mapsto [c:\lambda\mu d]$ on $\mathbb{P}^1$, the one of type (II) induces $[c:d] \mapsto [d:\lambda\mu c]$, and τ induces $\mathrm{id}_{\mathbb{P}^1}$. Hence, the image of $\mathrm{Aut}(\mathcal{S}^{L,L'}, \pi)$ in $\mathrm{Aut}(\mathbb{P}^1, \{t,s\})$ is $T_1 \rtimes \mathbb{Z}/2\mathbb{Z}$.

Let us compute $\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, \hat{\pi})$, its image in $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, \{t,s\})$ separately for each of the two cases $L = L'$ and L, L' not $\mathbf{k}$-isomorphic. We will use that $\mathcal{Q}^L_K \simeq \mathbb{P}^1_K \times \mathbb{P}^1_K$ for $K = LL'$, hence $(A_\lambda, B_\mu) \in \mathrm{Aut}_K(\mathcal{Q}^L_K, \hat{\pi})$ exactly if $\lambda, \mu \in K$.

(1) Suppose that $L = L'$. Then $\tau \in \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, \hat{\pi})$. An element $(A_\lambda, B_\mu) \in \mathrm{Aut}_L(\mathcal{Q}^L, \hat{\pi})$ is defined over $\mathbf{k}$ if and only $\lambda, \mu \in L$ and $A_\lambda = B_\mu^g$, which is equivalent to $\mu = \lambda^g$. In that case, $\lambda\mu = \lambda\lambda^g$, which is contained in $\mathbf{k}$. Therefore, the image $\mathrm{Aut}_{\mathbf{k}}(\mathcal{S}^{L,L'}, \pi)$ in $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, \{t,s\})$ is isomorphic to $D^{L,L}_{\mathbf{k}} \rtimes \mathbb{Z}/2$.

(2) Suppose that L and L' are not $\mathbf{k}$-isomorphic. Let $K = LL'$ and $\mathrm{Gal}(K/\mathbf{k}) \simeq \mathrm{Gal}(L/\mathbf{k}) \times \mathrm{Gal}(L'/\mathbf{k}) = \langle g \rangle \times \langle g' \rangle$. Let us compute $\mathrm{Aut}_K(\mathcal{Q}^L_K, \hat{\pi})$. Observe that we have $p_i = ([b_i:1],[b_i:1])$ for $i = 1,2$ and that we can no longer assume that they are equal to $([1:0],[0:1]), ([0:1],[1:0])$. However, the coordinate change given by

$$\gamma := \left(\begin{pmatrix} b_2 & b_1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ 1 & 1 \end{pmatrix} \right) \in \mathrm{PGL}_2(K) \times \mathrm{PGL}_2(K)$$

sends $([1:0],[0:1]), ([0:1],[1:0])$ onto p_1, p_2, respectively. One can compute that the Galois action on $\mathcal{Q}^L$ (see the proof of Lemma 3.2) induced by γ, namely $G' = \gamma^{-1} \circ \mathrm{Gal}(K/\mathbf{k}) \circ \gamma$, is given by

$$([u_0:u_1],[v_0:v_1]) \mapsto \begin{cases} ([v_1^g : v_0^g],[u_1^g : u_0^g]) \\ ([u_1^{g'} : u_0^{g'}],[v_1^{g'} : v_0^{g'}]). \end{cases}$$

Note that τ is G'-invariant and so it remains to study which (A_λ, A_μ) are G'-invariant. So (A_λ, A_μ) is defined over $\mathbf{k}$ for $\lambda, \mu \in K$ if and only if

$$(A_\lambda, A_\mu) = (A_\lambda, A_\mu)^g = (A_{(\mu^{-1})^g}, A_{(\lambda^{-1})^g})$$
$$(A_\lambda, A_\mu) = (A_\lambda, A_\mu)^{g'} = (A_{(\lambda^{-1})^{g'}}, A_{(\mu^{-1})^{g'}}).$$

Hence, the elements of $\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, \hat{\pi})$ are exactly those of the form $\gamma \circ (A_\lambda, A_\mu) \circ \gamma^{-1}$ with $\lambda, \mu \in K$ satisfying $\lambda = (\mu^{-1})^g$, $\mu = (\lambda^{-1})^g$ $\lambda = (\lambda^{-1})^{g'}$, $\mu = (\mu^{-1})^{g'}$.

Instead of computing the image of $\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, \hat{\pi})$ in $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1)$, we compute the image of $\gamma^{-1}\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L, \hat{\pi})\gamma$ (*i.e.* (A_λ, A_μ)) on $\mathrm{Aut}_K(\mathbb{P}^1)$ with the induced Galois action on $\mathbb{P}^1$, which is given by $([c:d])^g = [d^g : c^g]$ and $([c:d])^{g'} = [d^{g'} : c^{g'}]$. Again, (A_λ, A_μ) induces $[c:d] \mapsto [c:\lambda\mu d]$ or $[c:d] \mapsto [d:\lambda\mu c]$, and τ' induces $\mathrm{id}_{\mathbb{P}^1}$. We compute the possible $\delta = \lambda\mu$: On one hand we find

$$\lambda\mu = (\mu^{-1})^g(\lambda^{-1})^g = (\mu^{g'})^g(\lambda^{g'})^g = (\lambda\mu)^{gg'},$$

implying $\delta \in F$, where $\mathbf{k} \subset F \subset K$ with $\mathrm{Gal}(F/\mathbf{k}) = \langle gg' \rangle$. On the other hand, we also have

$$\lambda\mu = \lambda(\mu^{-1})^{g'} = \lambda\lambda^{g \cdot g'}$$

Hence, $D^{L,L'}_{\mathbf{k}}$ is conjugated to $\{\lambda\lambda^{gg'} \in F \mid \lambda \in K,\ \lambda\lambda^{g'} = 1\}$. □

In the lemma above, if L, L' are not $\mathbf{k}$-isomorphic, then $D^{L,L'}_{\mathbf{k}} \simeq \{N_{F/\mathbf{k}}(\lambda) \mid \lambda \in K,\ N_{K/L}(\lambda) = 1\}$, where $N_{F/\mathbf{k}}$ and $N_{K/L}$ are the field norms of $F/\mathbf{k}$ and K/L, respectively.

5. The conic fibration cases

In this section, we classify the rational conic fibrations $\pi\colon X \to \mathbb{P}^1$ that are $\mathrm{Aut}(X,\pi)$-Mori fibre spaces. Recall that π induces a homomorphism $\mathrm{Aut}(X,\pi) \to \mathrm{Aut}(\mathbb{P}^1)$ whose kernel we denote by $\mathrm{Aut}(X/\pi)$ and its $\mathbf{k}$-points by $\mathrm{Aut}_{\mathbf{k}}(X/\pi)$.

Recall from Lemma 2.8 that, for any Mori fibre space $\pi\colon X \longrightarrow \mathbb{P}^1$ such that X is rational, we have either $X \simeq \mathbb{F}_n$ for some $n \geqslant 0$ or $X \simeq \mathcal{S}^{L,L'}$ or X is isomorphic to a del Pezzo surface obtained by blowing up $\mathbb{P}^2$ in a point of degree 4. In the latter case, $\mathrm{Aut}(X,\pi)$ is finite by Lemma 2.10, so we do not look at it.

5.1. **Conic fibrations obtained by blowing up a Hirzebruch surface.** We study the rational conic fibrations $\pi\colon X \to \mathbb{P}^1$ that are $\mathrm{Aut}(X,\pi)$-Mori fibre spaces and for which there is a birational morphism $X \longrightarrow \mathbb{F}_n$ of conic fibrations for some $n \geqslant 0$.

Remark 5.1. Let $n \geqslant 1$ and denote by $\mathbf{k}[z_0,z_1]_n \subset \mathbf{k}[z_0,z_1]$ the vector space of homogeneous polynomials of degree n. In the coordinates from Example 2.5(1) the special section $S_{-n} \subset \mathbb{F}_n$ is given by $y_0 = 0$. We denote by $S_n \subset \mathbb{F}_n$ the section given by $y_1 = 0$. Since $S_n \cdot S_{-n} = 0$, we have $S_n \sim S_{-n} + nf$ and $S_n^2 = n$, where f is the class of a fibre. The automorphism group of $\mathbb{F}_n$ is

$$\mathrm{Aut}(\mathbb{F}_n) = \mathrm{Aut}(\mathbb{F}_n,\pi_n) \simeq V_{n+1} \rtimes \mathrm{GL}_2/\mu_n, \quad \mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_n) \simeq \mathbf{k}[z_0,z_1]_n \rtimes \mathrm{GL}_2(\mathbf{k})/\mu_n(\mathbf{k}),$$

where V_{n+1} is the canonical $\mathbf{k}$-structure on $\overline{\mathbf{k}}[z_0,z_1]_n$ and $\mu_n = \{\lambda \cdot \mathrm{id} \in \mathrm{GL}_2 \mid \lambda^n = 1\}$. The group $\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_n)$ acts on $\mathbb{F}_n$ by

$$[y_0 : y_1; z_0 : z_1] \mapsto [y_0 : P(z_0,z_1)y_0 + y_1; az_0 + bz_1 : cz_0 + dz_1],$$

and it has two orbits on $\mathbb{F}_n$, namely S_{-n} and $\mathbb{F}_n \backslash S_{-n}$.

Lemma 5.2. *Let $n \geqslant 0$ and $\eta\colon X \to \mathbb{F}_n$ be a birational morphism of conic fibrations that is not an isomorphism, and suppose that $\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\pi)$ contains an element permuting the components of at least one singular geometric fibre. Let $G_{\overline{\mathbf{k}}} \subset \mathrm{Aut}_{\overline{\mathbf{k}}}(X/\pi)$ be the subgroup of elements acting trivially on $\mathrm{NS}(X_{\overline{\mathbf{k}}})$.*

1. *If $G_{\overline{\mathbf{k}}}$ is non-trivial, there exists $N \geqslant 1$ and a birational morphism $X \to \mathbb{F}_N$ of conic fibrations blowing up $r \geqslant 1$ points $p_1,\dots,p_r$ contained in S_N such that $\sum_{i=1}^r \deg(p_i) = 2N$.*
2. *If $G_{\overline{\mathbf{k}}} = \{1\}$, then $\mathrm{Aut}_{\overline{\mathbf{k}}}(X/\pi) \simeq (\mathbb{Z}/2)^r$ for $r \in \{0,1,2\}$.*

Proof. The claim is proven in [5, Lemme 4.3.5] over $\mathbb{C}$ and its proof can be repeated word by word over any algebraically closed field. Over a perfect field $\mathbf{k}$ it suffices to show that curves contracted by the birational morphism $\nu\colon X_{\overline{\mathbf{k}}} \longrightarrow (\mathbb{F}_N)_{\overline{\mathbf{k}}}$ in (1) are already defined over $\mathbf{k}$. Since $N \geqslant 1$, the surface $X_{\overline{\mathbf{k}}}$ contains exactly two sections of negative self-intersection, namely the strict transforms $\tilde{S}_{-N}$ and $\tilde{S}_N$ of S_{-N} and S_N, respectively, and $\tilde{S}_{-N}^2 = \tilde{S}_N^2 = -N$, and every singular geometric fibre has two components, each intersecting either $\tilde{S}_{-N}$ or $\tilde{S}_N$. We now show that $\tilde{S}_{-N}$ and $\tilde{S}_N$ are both defined over $\mathbf{k}$, which will then imply that the curves contracted by η are defined over $\mathbf{k}$ and we are finished. The birational morphism $\eta\colon X \longrightarrow \mathbb{F}_n$ contracts exactly one component in each singular fibre. This implies that the strict transform $\tilde{S}_{-n}$ of $S_{-n} \subset \mathbb{F}_n$ has self-intersection $\leqslant -n$. If $n \geqslant 1$, then $\tilde{S}_{-n}$ is one of $\tilde{S}_N$ or $\tilde{S}_{-N}$ and hence both $\tilde{S}_N$ or $\tilde{S}_{-N}$ are defined over $\mathbf{k}$. If $n = 0$, then $\eta(\tilde{S}_{-N})$ and $\eta(\tilde{S}_N)$ are sections in $\mathbb{F}_0$ of ruling induced by η. If they are permuted by an element of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$, each fibre contains two points blown-up by η, which contradicts $X \longrightarrow \mathbb{P}^1$ being a conic fibration. It follows that $\eta(\tilde{S}_{-N})$ and $\eta(\tilde{S}_N)$ are both defined over $\mathbf{k}$ and hence $\tilde{S}_{-N}, \tilde{S}_N$ are defined over $\mathbf{k}$ as well. $\square$

Let us construct a special birational involution of $\mathbb{F}_n$, $n \geqslant 1$.

Example 5.3. Let $n \geqslant 1$. Let $p_1, \dots, p_r \in S_n \subset \mathbb{F}_n$ be points such that their geometric components are in pairwise distinct geometric fibres and $\sum_{i=1}^r \deg(p_i) = 2n$, and assume that $\pi_n(p_i) \neq [0:1], [1:0]$ for $i = 1, \dots, r$. Let $P_i \in \mathbf{k}[z_0, z_1]_{\deg(p_i)}$ be the polynomial defining $\pi(p_i) \in \mathbb{P}^1$ and define $P := P_1 \cdots P_r \in \mathbf{k}[z_0, z_1]_{2n}$. Then the map

$$\varphi\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_n,\ (y_1, z_1) \mapsto (P(z_1)/y_1, z_1)$$

is an involution preserving the fibration, whose base-points are $p_1, \dots, p_r$, that exchanges S_n and S_{-n} and contracts the fibres through $p_1, \dots, p_r$.

We call $\mu_n \subset T_1$ the subgroup of n^{th} roots of unity of the 1-dimensional standard torus T_1.

Lemma 5.4. *Let $n \geqslant 1$ and let $\eta\colon X \to \mathbb{F}_n$ be a birational morphism blowing up points $p_1, \dots, p_r \in S_n$ whose geometric components are on pairwise distinct geometric fibres and such that $\sum_{i=1}^r \deg(p_i) = 2n$. Then $\pi := \pi_n \eta\colon X \longrightarrow \mathbb{P}^1$ is a conic fibration that has exactly two $(-n)$-sections and the following properties hold.*

(1) There are split exact sequences

$$1 \to \mathrm{Aut}(X/\pi) \longrightarrow \mathrm{Aut}(X, \pi) \longrightarrow \mathrm{Aut}(\mathbb{P}^1, \Delta) \to 1$$
$$1 \to \mathrm{Aut}_{\mathbf{k}}(X/\pi) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X, \pi) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, \Delta) \to 1$$

where $\Delta \subset \mathbb{P}^1$ is the image of the singular fibres of $X/\mathbb{P}^1$.

(2) The action of $\mathrm{Aut}(X/\pi)$ on the two $(-n)$-sections induces split exact sequences

$$1 \to H \longrightarrow \mathrm{Aut}(X/\pi) \longrightarrow \mathbb{Z}/2 \to 1,$$
$$1 \to H(\mathbf{k}) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X/\pi) \longrightarrow \mathbb{Z}/2 \to 1$$

where $\eta H \eta^{-1} = \mathrm{Aut}(\mathbb{F}_n/\pi_n, S_n) \simeq T_1/\mu_n$ and $\eta H(\mathbf{k})\eta^{-1} \simeq \mathbf{k}^*/\mu_n(\mathbf{k})$, and $\mathbb{Z}/2 = \langle \eta^{-1}\varphi\eta \rangle$ with $\varphi\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_n$ the involution from Example 5.3.

(3) Any element of $\mathrm{Aut}_{\mathbf{k}}(X/\pi) \backslash H(\mathbf{k})$ is an involution fixing an irreducible double cover of $\mathbb{P}^1$ branched over Δ not intersecting S_{-n}.

(4) $\pi\colon X \longrightarrow \mathbb{P}^1$ is an $\mathrm{Aut}(X, \pi)$-Mori fibre space and an $\mathrm{Aut}_{\mathbf{k}}(X, \pi)$-Mori fibre space.

Proof. We denote by $\tilde{S}_n$ and $\tilde{S}_{-n}$ the strict transforms of the sections S_n and S_{-n} of $\mathbb{F}_n$ in X, which satisfy $\tilde{S}_n^2 = \tilde{S}_{-n}^2 = -n$ and which are the only (geometric) sections of negative self-intersection. The anti-canonical divisor of X is π-ample because the geometric components of the p_i are on pairwise distinct geometric fibres, thus $\pi\colon X \longrightarrow \mathbb{P}^1$ is a conic fibration with r singular fibres, each of whose geometric components intersects exactly one of the sections $\tilde{S}_n$ and $\tilde{S}_{-n}$.

(1) For any element $\alpha \in \mathrm{Aut}(\mathbb{P}^1, \Delta)$ there exists $\tilde{\alpha} \in \mathrm{Aut}(\mathbb{F}_n)$ preserving $\{p_1, \dots, p_r\}$, and we have $\eta^{-1}\tilde{\alpha}\eta \in \mathrm{Aut}(X, \pi)$. The same argument holds for the $\mathbf{k}$-points of these groups.

(2) Up to an element of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_n)$, we can assume that $\pi_n(p_i) \neq [1:0], [0:1]$ for $i = 1, \dots, r$. Then the birational involution $\varphi\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_n$ from Example 5.3 lifts to an element of $\mathrm{Aut}_{\mathbf{k}}(X/\pi)$ and exchanges $\tilde{S}_n$ and $\tilde{S}_{-n}$. It follows that the action of $\mathrm{Aut}(X/\pi)$ on $\{\tilde{S}_n, \tilde{S}_{-n}\}$ induces split exact sequences

$$1 \to H \longrightarrow \mathrm{Aut}(X/\pi) \longrightarrow \mathbb{Z}/2 \to 1, \quad \text{and} \quad 1 \to H(\mathbf{k}) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X/\pi) \longrightarrow \mathbb{Z}/2 \to 1.$$

Any element of H fixes $\tilde{S}_n$ and $\tilde{S}_{-n}$ pointwise, so $\eta H \eta^{-1}$ and $\eta H(\mathbf{k})\eta^{-1}$ are the subgroups of $\mathrm{Aut}(\mathbb{F}_n/\pi_n) \simeq V_{n+1} \rtimes T_1/\mu_n$ and $\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_n/\pi_n) \simeq \mathbf{k}[z_0, z_1]_n \rtimes \mathbf{k}^*/\mu_n(\mathbf{k})$, respectively, fixing S_n pointwise. It follows that $\eta H \eta^{-1} = T_1/\mu_n$ and $\eta H(\mathbf{k})\eta^{-1} = \mathbf{k}^*/\mu_n(\mathbf{k})$.

(4) The fact that the element $\eta^{-1}\varphi\eta \in \mathrm{Aut}_{\mathbf{k}}(X/\pi)$ exchanges the components of every singular geometric fibre implies that $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X,\pi)} = 1$. It follows that $X/\mathbb{P}^1$ is an $\mathrm{Aut}_{\mathbf{k}}(X,\pi)$-Mori fibre space and in particular an $\mathrm{Aut}(X,\pi)$-Mori fibre space.

(3) For any $\lambda \in \mathbf{k}^*$ the map

$$(\lambda, \varphi)\colon (y_1, z_1) \dashrightarrow ({}^{\lambda^n P(z_1)}/_{y_1}, z_1)$$

is a birational involution of $\mathbb{F}_n$ and fixes the curve $y_1^2 - \lambda^n P(z_0, z_1) y_0^2 = 0$, which is a double cover of $\mathbb{P}^1$ branched over Δ and does not intersect the section S_{-n}. □

Lemma 5.5. *Let $n \geqslant 1$ and $\eta\colon X \to \mathbb{F}_n$ be a birational morphism blowing up points $p_1, \ldots, p_r \in S_n$ whose geometric components are on pairwise distinct geometric fibres and such that $\sum_{i=1}^r \deg(p_i) = 2n$. Let $\pi = \pi_n\eta\colon X \longrightarrow \mathbb{P}^1$ be the induced conic fibration on X.*

(1) If $n = 1$, then X is a del Pezzo surface of degree 6 as in 1(1) or 1(3) and $\mathrm{Aut}(X,\pi) \subsetneq \mathrm{Aut}(X)$. Moreover, $\mathrm{Aut}_{\mathbf{k}}(X,\pi) \subsetneq \mathrm{Aut}_{\mathbf{k}}(X)$ if X is as in 1(1) and $\mathrm{Aut}_{\mathbf{k}}(X,\pi) = \mathrm{Aut}_{\mathbf{k}}(X)$ if X is as in 1(3).

(2) If $n \geqslant 2$, then $\mathrm{Aut}(X,\pi) = \mathrm{Aut}(X)$.

Proof. (1) For $n = 1$, the conic fibration $X/\mathbb{P}^1$ has two (-1)-sections and X is a del Pezzo surface of degree 6 as in Figure 1(1) or Figure 1(3). Lemma 4.1(2) applied to $X_{\overline{\mathbf{k}}}$ implies that $\mathrm{Aut}(X)$ contains an element inducing a rotation of order 6 on the hexagon of X, which is not contained in $\mathrm{Aut}(X,\pi)$. The same argument implies that $\mathrm{Aut}_{\mathbf{k}}(X,\pi) \subsetneq \mathrm{Aut}_{\mathbf{k}}(X)$ if X is a del Pezzo surface of degree 6 as in 1(1). However, in the case of Figure 1(3), any element of $\mathrm{Aut}_{\mathbf{k}}(X)$ preserves the fibration by Lemma 4.9(4).

(2) If $n \geqslant 2$, X contains exactly two $(-n)$-sections $\tilde{S}_n$ and $\tilde{S}_{-n}$, which are the strict transforms of S_n and S_{-n}. Thus the class $\tilde{S}_n + \tilde{S}_{-n}$ in $\mathrm{NS}(X_{\overline{\mathbf{k}}})$ is $\mathrm{Aut}_{\overline{\mathbf{k}}}(X)$-invariant, hence $K_X + (\tilde{S}_n + \tilde{S}_{-n}) = -2f$ is $\mathrm{Aut}_{\overline{\mathbf{k}}}(X)$-invariant as well. It follows that $\mathrm{Aut}(X) = \mathrm{Aut}(X,\pi)$. □

If two conic fibrations as in Lemma 5.4 are isomorphic, they both have a birational morphism to the same Hirzebruch surface $\mathbb{F}_n$.

Lemma 5.6. *For any fixed $n \geqslant 1$, two conic fibrations as in Lemma 5.4 are isomorphic if and only if the points on $\mathbb{P}^1$ are the same, up to an element of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1)$.*

Proof. Any element of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1)$ lifts to an element of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_n)$, so two such conic fibrations are isomorphic, if and only if the points on the section S_n are the same, up to an element of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_n)$. This means that their images on $\mathbb{P}^1$ are the same, up to an element of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1)$. □

5.2. **Conic fibrations obtained by blowing up a del Pezzo surface.** Let $L = \mathbf{k}(a_1)$ and $L' = \mathbf{k}(b_1)$ be quadratic extensions of $\mathbf{k}$. In this section, we consider rational conic fibrations $\pi\colon X \longrightarrow \mathbb{P}^1$ for which there is a birational morphism $\eta\colon X/\mathbb{P}^1 \longrightarrow \mathcal{S}^{L,L'}/\mathbb{P}^1$ of conic fibrations, where $\pi_{\mathcal{S}^{L,L'}}\colon \mathcal{S}^{L,L'} \longrightarrow \mathbb{P}^1$ is the Mori fibre space from Example 2.5(2). We have described the fibration $S^{L,L'} \longrightarrow \mathbb{P}^1$ in Section 4.6.

Recall from Lemma 4.11(1) and Lemma 4.12(1) that there is a birational morphism $\nu\colon \mathcal{S}^{L,L'} \longrightarrow \mathcal{Q}^L$ contracting a curve E onto a point p' of degree 2 with splitting field L'.

Remark 5.7. Let $p \in E \subset \mathcal{S}^{L,L'}$ be a point whose geometric components are in distinct smooth geometric fibres of $\mathcal{S}^{L,L'}/\mathbb{P}^1$. Any element of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ exchanges or preserves the geometric components of the point $\eta(E)$ and hence of the curve E, and this implies that $\deg(p)$ is even and each geometric component of E contains $\frac{\deg(p)}{2}$ geometric components of p.

We now show an analogue of Lemma 5.2, that we prove similarly to [5, Lemme 4.3.5].

Lemma 5.8. *Let $\eta\colon X \longrightarrow \mathcal{S}^{L,L'}$ be a birational morphism of conic fibrations that is not an isomorphism, and suppose that $\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\pi)$ contains an element exchanging the components of at least one singular geometric fibre. Let $G_{\overline{\mathbf{k}}} \subset \mathrm{Aut}_{\overline{\mathbf{k}}}(X/\pi)$ be the subgroup acting trivially on $\mathrm{NS}(X_{\overline{\mathbf{k}}})$.*

(1) If $G_{\overline{\mathbf{k}}}$ is non-trivial, then η is the blow-up of $r \geqslant 1$ points contained in $E \subset \mathcal{S}^{L,L'}$ whose geometric components are on pairwise distinct smooth geometric fibres, and each geometric component of E contains half of the geometric components of each point.

(2) If $G_{\overline{\mathbf{k}}} = \{1\}$, then $\mathrm{Aut}_{\mathbf{k}}(X/\pi) \simeq (\mathbb{Z}/2)^r$ for $r \in \{0,1,2\}$.

Proof. (1) Suppose that $G_{\overline{\mathbf{k}}}$ is nontrivial. It preserves the geometric components of the singular fibres, so η is $G_{\overline{\mathbf{k}}}$-equivariant and $R := \eta G_{\overline{\mathbf{k}}}\eta^{-1} \subset \mathrm{Aut}_{\overline{\mathbf{k}}}(\mathcal{S}^{L,L'}_{\overline{\mathbf{k}}}/\pi_{\mathcal{S}^{L,L'}})$. The group R fixes the geometric components of E pointwise. Since $R \subset \mathrm{PGL}_2(\overline{\mathbf{k}}(x))$ and since it is non-trivial, it fixes no other sections of $\mathcal{S}^{L,L'}_{\overline{\mathbf{k}}}/\mathbb{P}^1$. So, $G_{\overline{\mathbf{k}}}$ fixes the geometric components of the strict transform $\tilde{E} \subset X$ of E and no other sections of $X_{\overline{\mathbf{k}}}/\mathbb{P}^1_{\overline{\mathbf{k}}}$. Moreover, $\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\pi)$ contains an element exchanging the components of at least one singular geometric fibre, so it follows that each geometric component of $\tilde{E}$ intersects exactly one component of each geometric singular fibre. In particular, the points blown-up by η are contained in E. The hypothesis that $-K_X$ is π-ample implies that the geometric components of the blown-up points are on distinct geometric components of smooth fibres. The remaining claim follows from Remark 5.7.

(2) If $G_{\overline{\mathbf{k}}}$ is trivial, then every non-trivial element of $\mathrm{Aut}_{\overline{\mathbf{k}}}(X/\pi)$ is an involution and the claim follows from the fact that $\mathrm{Aut}_{\overline{\mathbf{k}}}(X/\pi) \subset \mathrm{PGL}_2(\overline{\mathbf{k}}(x))$. □

Example 5.9. Let us construct a special birational involution of $\varphi_{L,L'}$ of $\mathcal{S}^{L,L'}$ that preserves the fibration $\mathcal{S}^{L,L'} \longrightarrow \mathbb{P}^1$ and induces the identity on $\mathbb{P}^1$.

Let E_1, E_2 be the geometric components of E. If g' is the generator of $\mathrm{Gal}(L'/\mathbf{k})$, then $E_2^g = E_1$. Let $p_1, \ldots, p_r \in E \subset \mathcal{S}^{L,L'}$ be points whose geometric components are on pairwise distinct smooth geometric fibres. We now construct an involution φ of $\mathcal{S}^{L,L'}$ whose base-points are $p_1, \ldots, p_r$ and which exchanges E_1 and E_2. For $i = 1,2$, let $P_i \in L[x,y]$ be homogeneous polynomials defining the set of components of the $p_1, \ldots, p_r$ contained in E_i. Consider a birational morphism $\mathcal{S}^{L,L'} \longrightarrow \mathcal{Q}^L$ that contracts E, and consider the model of $\mathcal{Q}^L$ that is a $\mathbf{k}$-structure on $\mathbb{P}^1_L \times \mathbb{P}^1_L$.

- If L and L' are $\mathbf{k}$-isomorphic, we can assume that the images of E_1 and E_2 are respectively $([1:0],[0:1])$ and $([0:1],[1:0])$, by Lemma 3.6(1). We define

$$\tilde{\varphi}_{L,L}\colon ([u_0:u_1],[v_0:v_1]) \mapsto \\ ([v_0P_1(u_0v_0,u_1v_1) : v_1P_2(u_0v_0,u_1v_1)],[u_0P_2(u_0v_0,u_1v_1) : u_1P_1(u_0v_0,u_1v_1)]).$$

- If L and L' are not $\mathbf{k}$-isomorphic, we write $L' = \mathbf{k}(b_1)$. By Lemma 3.3(3a), we can assume that the images of E_1, E_2 are $([b_1:1],[b_1:1]), ([b_2:1],[b_2:1])$. To compute $\varphi_{L,L'}$, we simply conjugate $\varphi_{L,L}$ over $\overline{\mathbf{k}}$ with

$$\gamma := \left(\begin{pmatrix} b_2 & b_1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ 1 & 1 \end{pmatrix}\right) \in \mathrm{PGL}_2(\overline{\mathbf{k}}) \times \mathrm{PGL}_2(\overline{\mathbf{k}})$$

This yields the following form of $\varphi_{L,L'}$

$$\tilde{\varphi}_{L,L'}\colon ([u_0:u_1],[v_0:v_1]) \mapsto ([v_0U + v_1V : v_0W - v_1U],[u_0U + u_1V : u_0W - u_1U])$$

where

$$U := b_2P_1(t,s) - b_1P_2(t,s),\ V := b_1^2P_2(t,s) - b_2^2P_1(t,s),\ W := P_1(t,s) - P_2(t,s)$$

with

$$t := (u_0 - b_1u_1)(v_0 - b_2v_1),\ s := (u_0 - b_2v_1)(v_0 - b_1v_1).$$

In both cases, $\tilde{\varphi}_{L,L'}$ commutes with $\mathrm{Gal}(L/\mathbf{k})$ and $\mathrm{Gal}(L'/\mathbf{k})$ and it is an involution. Moreover, it preserves the image of the fibration $\mathcal{S}^{L,L'} \longrightarrow \mathbb{P}^1$ in $\mathcal{Q}^L$ and induces the identity map on $\mathbb{P}^1$. The base-locus of $\tilde{\varphi}_{L,L'}$ in $\mathcal{Q}^L$ is the image of E, and $\tilde{\varphi}_{L,L'}$ contracts the image of the fibres of $\mathcal{S}^{L,L'} \longrightarrow \mathbb{P}^1$ given by $P_1P_2 = 0$. It follows that $\tilde{\varphi}_{L,L'}$ lifts to a birational involution $\varphi_{L,L'}$ not defined in $p_1, \dots, p_r$.

Lemma 5.10. *Let $\eta\colon X \to \mathcal{S}^{L,L'}$ be the blow-up up of points $p_1, \dots, p_r \in E$, $r \geqslant 1$, whose geometric components are on pairwise distinct smooth geometric fibres. Then $\pi := \pi_{\mathcal{S}}\eta\colon X \longrightarrow \mathbb{P}^1$ is a conic fibration and $\deg(p_i)$ is even and each geometric component of E contains $\frac{\deg(p_i)}{2}$ geometric components for $i = 1, \dots, r$. Moreover, the following hold.*

1. *The action of $\mathrm{Aut}(X,\pi)$ on $\mathbb{P}^1$ induces the exact sequence*
$$1 \to \mathrm{Aut}(X/\pi) \longrightarrow \mathrm{Aut}(X,\pi) \longrightarrow \mathrm{Aut}(\mathbb{P}^1,\Delta) \to 1$$
$$1 \to \mathrm{Aut}_{\mathbf{k}}(X/\pi) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X,\pi) \longrightarrow (D_{\mathbf{k}}^{L,L'} \rtimes \mathbb{Z}/2) \cap \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1,\Delta) \to 1$$
where $D_{\mathbf{k}}^{L,L'} \rtimes \mathbb{Z}/2$ is the image of $\mathrm{Aut}_{\mathbf{k}}(\mathcal{S}^{L,L'},\pi)$ in $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1)$, see Lemma 4.15, and $\Delta \subset \mathbb{P}^1$ is the image of the singular fibres of X.
2. *The $\mathrm{Aut}(X/\pi)$-action on the components of the strict transform of E induces the split exact sequences*
$$1 \to H \longrightarrow \mathrm{Aut}(X/\pi) \longrightarrow \mathbb{Z}/2 \to 1,$$
$$1 \to H(\mathbf{k}) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X/\pi) \longrightarrow \mathbb{Z}/2 \to 1$$
with $\eta H \eta^{-1} = \mathrm{SO}^{L,L'}$ from Lemma 4.14 and $\mathbb{Z}/2$ is generated by the involution $\varphi_{L,L'}\colon \mathcal{S}^{L,L'} \dashrightarrow \mathcal{S}^{L,L'}$ from Example 5.9.
3. *Any element of $\mathrm{Aut}_{\mathbf{k}}(X/\pi)\backslash H(\mathbf{k})$ is an involution fixing an irreducible double cover of $\mathbb{P}^1$ branched over Δ.*
4. *$\pi\colon X \longrightarrow \mathbb{P}^1$ is an $\mathrm{Aut}(X,\pi)$-Mori fibre space and an $\mathrm{Aut}_{\mathbf{k}}(X,\pi)$-Mori fibre space.*

Proof. The first claim follows from Remark 5.7 and the sequences in (1) are exact by Lemma 4.14.

(2) Consider the involution $\varphi_{L,L'}\colon \mathcal{S}^{L,L'} \dashrightarrow \mathcal{S}^{L,L'}$ from Example 5.9 whose base-points are $p_1, \dots, p_r$ and that exchanges the geometric components of E. Then $\hat{\varphi}_{L,L'} := \eta^{-1}\varphi_{L,L'}\eta$ is contained in $\mathrm{Aut}_{\mathbf{k}}(X/\pi)$ and exchanges the geometric components of the strict transform $\tilde{E}$ of E. In particular, the $\mathrm{Aut}(X/\pi)$-action on the set of geometric components of $\tilde{E}$ induces split exact sequences as claimed. The groups H and $H(\mathbf{k})$ are respectively conjugate by η to the subgroups of $\mathrm{Aut}(\mathcal{S}^{L,L'}/\pi_{\mathcal{S}})$ and $\mathrm{Aut}_{\mathbf{k}}(\mathcal{S}^{L,L'}/\pi_{\mathcal{S}})$ preserving the geometric components of E, which are $\mathrm{SO}^{L,L'}$ and $\mathrm{SO}^{L,L'}(\mathbf{k})$ by Lemma 4.14.

(3) It is enough to show that this is already the case for any element in $\mathrm{Aut}_{\overline{\mathbf{k}}}(X/\pi)\backslash H(\overline{\mathbf{k}})$. Indeed, we have $\mathrm{Aut}_{\overline{\mathbf{k}}}(X/\pi) \simeq H(\overline{\mathbf{k}}) \rtimes \mathbb{Z}/2$, and any element of $\mathrm{Aut}_{\overline{\mathbf{k}}}(X/\pi)\backslash H(\overline{\mathbf{k}})$ is of the form $(\eta^{-1}\alpha\eta, \hat{\varphi}_{L,L'})$, where $\alpha := (a, a^{-1}) \in \mathrm{SO}^{L,L'}(\overline{\mathbf{k}})$. Using Example 5.9, we compute that $(\eta^{-1}\alpha\eta, \hat{\varphi}_{L,L'})$ is an involution. Its fixed $\overline{\mathbf{k}}$-curve in $\mathcal{Q}^L_{\overline{\mathbf{k}}}$ is given by

$$au_0v_1P_2(u_0v_0, u_1v_1) - u_1v_0P_1(u_0v_0, u_1v_1) = 0$$

which lifts to the desired curve on $X_{\overline{\mathbf{k}}}$.

(4) The involution $\hat{\varphi}$ exchanges the geometric components of all singular fibres and hence $X \longrightarrow \mathbb{P}^1$ is a $\mathrm{Aut}(X, \mathbb{P}^1)$-Mori fibre space and an $\mathrm{Aut}_{\mathbf{k}}(X, \mathbb{P}^1)$-Mori fibre space. □

Lemma 5.11. *Let $\eta\colon X \to \mathcal{S}^{L,L'}$ be the blow-up up of points $p_1, \dots, p_r \in E$, $r \geqslant 1$, whose geometric components are on pairwise distinct smooth geometric fibres. Then $\mathrm{Aut}(X, \pi) = \mathrm{Aut}(X)$.*

Proof. By Remark 5.7, each of the components of E contains half the geometric components of each p_i. It follows that $n := \frac{1}{2}\sum_{i=1}^{r} \deg(p_i) \in \mathbb{Z}$ and $n \geqslant 1$. For $i = 1, \dots, r$, let E_i be the exceptional divisor of p_i and let f be a general fibre of X and $\tilde{E}$ the strict transform of E. We have $K_{\mathcal{S}} = -2f - E$ and hence $K_X = -2f - \pi^*E + E_1 + \cdots + E_r = -2f - \tilde{E}$. The curve $\tilde{E}$ is the unique curve in X with self-intersection $\tilde{E}^2 = -2(1+n) \leqslant -4$ and hence it is $\mathrm{Aut}(X)$-invariant. In particular, $K_X + \tilde{E} = -2f$ is $\mathrm{Aut}(X)$-invariant. It follows that $\mathrm{Aut}(X) = \mathrm{Aut}(X, \pi)$. □

Lemma 5.12. *Two conic fibrations as in Lemma 5.10 are isomorphic as conic fibrations if and only if the points on $\mathbb{P}^1$ are the same, up to an element of $D_{\mathbf{k}}^{L,L'} \rtimes \mathbb{Z}/2$, which is the image of $\mathrm{Aut}_{\mathbf{k}}(\mathcal{S}^{L,L'}, \pi)$ in $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1)$ (see Lemma 4.15).*

Proof. Let $X \longrightarrow \mathcal{S}^{L,L'}$ and $X' \longrightarrow \mathcal{S}^{L,L'}$ be such conic fibrations obtained by blowing up $p_1, \dots, p_r \subset E$ and $p'_1, \dots, p'_s \subset E$, respectively, and suppose that they are isomorphic as conic fibrations. Then this isomorphism sends the singular fibres of X onto the ones of X', and hence descends to an automorphism of $\mathbb{P}^1$ that sends the images of the p_i onto the images of the p'_i.

On the other hand, given an automorphism α of $\mathbb{P}^1$ contained in $D_{\mathbf{k}}^{L,L'} \rtimes \mathbb{Z}/2$, we know by Lemma 5.10 there exists an automorphism ψ of X that induces α on $\mathbb{P}^1$. If α sends the p_i onto the p'_i, then either ψ or $\psi \circ \varphi$ sends the p_i onto the p'_i, where φ is the generator of $\mathbb{Z}/2 \subset \mathrm{Aut}_{\mathbf{k}}(X/\pi)$ in Lemma 5.10(2) exchanging the components of the singular fibres. □

6. The proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Lemma 6.1. *Consider a birational morphism of conic fibrations $X \longrightarrow \mathbb{F}_n$ for some $n \geqslant 0$, and suppose that $X/\mathbb{P}^1$ has at most two singular geometric fibres. If there is an element of $\mathrm{Aut}_{\overline{\mathbf{k}}}(X, \pi)$ that permutes the components of at least one singular geometric fibre, then it has exactly two singular geometric fibres and X is a del Pezzo surface of degree 6.*

Proof. Denote by $\eta\colon X \longrightarrow \mathbb{F}_n$ the birational morphism. Let $\tilde{S}_{-n} \subset X$ be the strict transform of the section $S_{-n} \subset \mathbb{F}_n$. Then $\tilde{S}_{-n}^2 \in \{-n, -n-1, -n-2\}$. Let $\alpha \in \mathrm{Aut}_{\overline{\mathbf{k}}}(X, \pi)$ be an element that permutes the components of at least one singular geometric fibre f_0. Then $\tilde{S} := \alpha(\tilde{S}_{-n})$ is a section of $\eta \times \mathrm{id}\colon X_{\overline{\mathbf{k}}} \longrightarrow \mathbb{P}^1_{\overline{\mathbf{k}}}$ of self-intersection $\tilde{S}^2 = \tilde{S}_{-n}^2$, and it intersects the other component of f_0. It follows that $S := \eta(\tilde{S}) \subset \mathbb{F}_n$ is a section of self-intersection $S^2 \in \{-n+2, -n+1, -n\}$, depending on how many of the points blown up by η are contained in S_{-n}. Since $S^2 \geqslant 0$, we have $n \leqslant 2$. If $n = 2$, we have $S^2 = 0$ and hence $S \sim S_{-2} + f$, which means that $S \cdot S_{-2} = -1$, which is impossible. It follows that $n = 0$ or $n = 1$, and so X is a del Pezzo surface of degree 6 or 7. In the latter case, no element of $\mathrm{Aut}_{\overline{\mathbf{k}}}(X, \pi)$ permutes the components of the singular fibre, hence X is a del Pezzo surface of degree 6. □

Lemma 6.2. *Let $\pi\colon X \longrightarrow \mathbb{P}^1$ be a $\mathrm{Aut}(X,\pi)$-Mori fibre space with at least three singular geometric fibres and suppose that there is a birational morphism of conic fibrations $X \longrightarrow Y$, where $Y = \mathbb{F}_n$ for some $n \geqslant 0$ or $Y = \mathcal{S}^{L,L'}$, and that $\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\pi)$ is infinite. Then the pair $(X, \mathrm{Aut}(X))$ is as in Theorem 1.1(6).*

Proof. The hypothesis that X is an $\mathrm{Aut}(X,\pi)$-Mori fibre space implies that $\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\pi)$ contains an element permuting the components of a singular geometric fibre. Moreover, $X/\mathbb{P}^1$ has at least three singular geometric fibres, the image of the homomorphism $\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\pi) \longrightarrow \mathrm{Aut}_{\overline{\mathbf{k}}}(\mathbb{P}^1)$ is finite and hence the kernel $\mathrm{Aut}_{\overline{\mathbf{k}}}(X/\pi)$ is infinite.

First, suppose that $Y = \mathbb{F}_n$. Since $X/\mathbb{P}^1$ has singular fibres, η is not an isomorphism. Lemma 5.2 and the fact that $\mathrm{Aut}_{\overline{\mathbf{k}}}(X/\pi)$ is infinite imply that there exists $N \geqslant 1$ and a birational morphism $X \longrightarrow \mathbb{F}_N$ that blows up $p_1, \ldots, p_r \in S_N \subset \mathbb{F}_N$ whose geometric components are in distinct geometric fibres and such that $\sum_{i=1}^r \deg(p_i) = 2N$. Because π has at least three singular geometric fibres, Lemma 5.5(1) implies that $N \geqslant 2$, and now Lemma 5.5(2) implies that $\mathrm{Aut}(X,\pi) = \mathrm{Aut}(X)$. Lemma 5.4(1–2) implies that $(X, \mathrm{Aut}(X))$ is as in Theorem 1.1(6a).

Now, suppose that $Y = \mathcal{S}^{L,L'}$. Since $X/\mathbb{P}^1$ has at least three singular fibres, η is not an isomorphism. Since $\mathrm{Aut}_{\overline{\mathbf{k}}}(X/\pi)$ is infinite, Lemma 5.8 implies that η blows up points $p_1, \ldots, p_r \in E$ whose geometric components are on distinct smooth geometric fibres, and Remark 5.7 implies that they are all of even degree and each geometric component of E contains half the geometric components of each p_i. Lemma 5.11 implies that $\mathrm{Aut}(X,\pi) = \mathrm{Aut}(X)$. Lemma 5.10 and the description of $D_{\mathbf{k}}^{L,L'}$ in Lemma 4.15 imply that the pair $(X, \mathrm{Aut}(X))$ is as in Theorem 1.1(6b). □

Proof of Theorem 1.1. By Proposition 2.13, there is a G-equivariant birational map $\mathbb{P}^2 \dashrightarrow X$ to a G-Mori fibre space $\pi\colon X \longrightarrow B$ that is one of the following:

- B is a point and $X \simeq \mathbb{P}^2$ or X is a del Pezzo surface of degree 6 or 8,
- $B = \mathbb{P}^1$ and there is a (perhaps non-equivariant) birational morphism of conic fibrations $X \longrightarrow Y$ with $Y = \mathbb{F}_n$ for some $n \geqslant 0$ or $Y = \mathcal{S}^{L,L'}$.

By Lemma 2.14, it suffices to look at the case $G = \mathrm{Aut}(X)$ or $G = \mathrm{Aut}(X,\pi)$, respectively. The pair $(\mathbb{P}^2, \mathrm{Aut}(\mathbb{P}^2))$ is the one in Theorem 1.1(1).

If X is a del Pezzo surface of degree 8, then X is isomorphic to $\mathbb{F}_0$, to $\mathbb{F}_1$ or to $\mathcal{Q}^L$ for some quadratic extension $L/\mathbf{k}$ by Lemma 3.2(1). However, $\mathbb{F}_1$ has a unique (-1)-curve, which is hence $\mathrm{Aut}(\mathbb{F}_1)$-invariant and its contraction conjugates $\mathrm{Aut}(\mathbb{F}_1)$ to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$. It follows that $X = \mathcal{Q}^L$ or $X = \mathbb{F}_0$, *i.e.* the pair $(X, \mathrm{Aut}(X))$ is as in Theorem 1.1(2)–(3).

If X is a del Pezzo surface of degree 6, the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action on the hexagon of X is one of the actions in Figure 1(1)–(9). Lemma 4.1(2–3) applied to $X_{\overline{\mathbf{k}}}$ yields that $\mathrm{rk}\,\mathrm{NS}(X_{\overline{\mathbf{k}}})^{\mathrm{Aut}_{\overline{\mathbf{k}}}(X_{\overline{\mathbf{k}}})} = 1$ and that the action of $\mathrm{Aut}_{\overline{\mathbf{k}}}(X)$ on $\mathrm{NS}(X_{\overline{\mathbf{k}}})$ induces a split exact sequence

$$1 \longrightarrow (\overline{\mathbf{k}}^*)^2 \longrightarrow \mathrm{Aut}_{\overline{\mathbf{k}}}(X) \longrightarrow \mathrm{Sym}_3 \times \mathbb{Z}/2 \longrightarrow 1.$$

If the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action is as in Figure 1(7)and (9), Lemma 4.6 and Lemma 4.7 imply that the pair $(X, \mathrm{Aut}_{\mathbf{k}}(X))$ is as in Theorem 1.1(5a).

If the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action is as in Figure 1(2)–3) and (5), then Lemma 4.11 and Lemma 4.9 and Lemma 4.12 imply that the pair $(X, \mathrm{Aut}(X))$ is as in Theorem 1.1(5c).

If the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action is as in Figure 1(1), Lemma 4.1 implies that $(X, \mathrm{Aut}(X))$ is as in Theorem 1.1(5(b)i).

If the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action is as in Figure 1(4), Lemma 4.10 implies that $(X, \mathrm{Aut}(X))$ is as in Theorem 1.1(5(b)ii).

If the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action is as in Figure 1(6), Lemma 4.2 implies that $(X, \mathrm{Aut}(X))$ is as in Theorem 1.1(5(b)iii).

If the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action is as in Figure 1(8), Lemma 4.3 implies that $(X, \mathrm{Aut}(X))$ is as in Theorem 1.1(5(b)iv).

Suppose that X admits a conic fibration $\pi\colon X \longrightarrow \mathbb{P}^1$ that is an $\mathrm{Aut}(X,\pi)$-Mori fibre space and there is a birational morphism $\eta\colon X \longrightarrow Y$ where $Y = \mathbb{F}_n$ for some $n \geqslant 0$ or $Y = \mathcal{S}^{L,L'}$.

First, suppose that η is an isomorphism. If $X \overset{\eta}{\simeq} Y = \mathbb{F}_n$, recall that $\mathbb{F}_0$ and $\mathbb{F}_1$ have already been discussed above, and that the family $\mathrm{Aut}(\mathbb{F}_n)$, $n \geqslant 2$ is the family in Theorem 1.1(4), see Remark 5.1. If $X \overset{\eta}{\simeq} Y = \mathcal{S}^{L,L'}$, then $\mathrm{Aut}(\mathcal{S}^{L,L'},\pi) \subseteq \mathrm{Aut}(\mathcal{S}^{L,L'})$, and the pair $(\mathcal{S}^{L,L'}, \mathrm{Aut}(\mathcal{S}^{L,L'}))$ is as in Theorem 1.1(5c) by Lemma 4.11.

Now, suppose that η is not an isomorphism. Since $\pi\colon X \longrightarrow \mathbb{P}^1$ is an $\mathrm{Aut}(X,\pi)$-Mori fibre space, there is an element of $\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\pi)$ that permutes the components of at least one singular geometric fibre. If $X/\mathbb{P}^1$ has at most two singular fibres, then the fact that η is not an isomorphism implies that $Y = \mathbb{F}_n$, and Lemma 6.1 implies that X is a del Pezzo surface of degree 6. Then $\mathrm{Aut}(X,\pi) \subseteq \mathrm{Aut}(X)$ and we have already discussed the pair $(X, \mathrm{Aut}(X))$ above. If $X/\mathbb{P}^1$ has at least three singular fibres, recall that $\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\pi)$ is infinite by hypothesis, and now Lemma 6.2 implies that the pair $(X, \mathrm{Aut}(X))$ is as in Theorem 1.1(6). $\square$

7. Classifying maximal algebraic subgroups up to conjugacy

In this section we classify up to conjugacy and up to inclusion the maximal infinite algebraic subgroups of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$. For this, we first need to introduce the so-called Sarkisov program. As before, $\mathbf{k}$ is a perfect field throughout the section.

7.1. **The equivariant Sarkisov program.** The Sarkisov program is an algorithmic way to decompose birational maps between Mori fibre spaces into nice elementary birational maps between Mori fibre spaces. In dimension 2, it is classical and treated exhaustively in [19], and from a more modern point of view in [22]. In dimension 3, it is developed in [11] over algebraically closed fields of characteristic zero. A non-algorithmic generalisation to any dimension $\geqslant 2$ is given in [18] over $\mathbb{C}$.

For surfaces, the Sarkisov program over $\mathbf{k}$ is the $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-equivariant classical Sarkisov program over $\overline{\mathbf{k}}$. For an affine algebraic group G, we can consider two equivariant Sarkisov programs:

- The $G(\mathbf{k})$-equivariant Sarkisov program over $\mathbf{k}$; the links are $G(\mathbf{k})$-equivariant birational maps between $G(\mathbf{k})$-Mori fibre spaces. If $G = \mathrm{Aut}(X)$ is one of the groups from Theorem 1.1, it is the tool to give us the conjugacy class of $G(\mathbf{k})$ inside $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$.
- The G-equivariant Sarkisov program is the $G_{\overline{\mathbf{k}}} \times \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-equivariant Sarkisov program over $\overline{\mathbf{k}}$; the links are G-equivariant birational maps between G-Mori fibre spaces. If $G = \mathrm{Aut}(X)$ is one of the groups from Theorem 1.1, it is the tool to give us the morphisms $G \longrightarrow \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ up to conjugation by an element of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$.

As part of Theorem 1.2, we will prove that these two classifications are not the same if $\mathbf{k}$ has an extension of degree 2 or 3.

Over $\mathbb{C}$ and for connected algebraic groups G, the G-equivariant Sarkisov program in dimension $\geqslant 2$ is developed in [17].

Definition 7.1. Let G be an affine algebraic group. We now define $G(\mathbf{k})$-equivariant Sarkisov links. The notion of G-equivariant Sarkisov links is defined analogously by replacing $G(\mathbf{k})$ with G, bearing that by G-orbit we mean a $G_{\overline{\mathbf{k}}} \times \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbit.

A $G(\mathbf{k})$-*equivariant Sarkisov link* (or simply $G(\mathbf{k})$-*equivariant link*) is a $G(\mathbf{k})$-equivariant birational map $\varphi\colon X \dashrightarrow X'$ between $G(\mathbf{k})$-Mori fibre spaces $\pi\colon X \longrightarrow B$ and $\pi'\colon X' \longrightarrow B'$ that is one of the following:

$$\begin{tikzcd} & X' \arrow[dr, "\pi'"] & \\ X \arrow[ur, dashed, "\varphi"] \arrow[dr, "\pi"'] & & B' \arrow[dl] \\ & B & \end{tikzcd}$$

type I

$$\begin{tikzcd} & Y \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\ X \arrow[rr, dashed, "\varphi"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & B = B' & \end{tikzcd}$$

type II

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\pi"] & \\ X' \arrow[dr, "\pi'"'] & & B \arrow[dl] \\ & B' & \end{tikzcd}$$

type III

$$\begin{tikzcd} X \arrow[rr, "\simeq", "\varphi"'] \arrow[d, "\pi"] & & X' \arrow[d, "\pi'"] \\ B \arrow[dr] & & B' \arrow[dl] \\ & * & \end{tikzcd}$$

type IV

(type I) B is a point, B' is a curve, $\varphi^{-1}\colon X' \longrightarrow X$ is the contraction of the $G(\mathbf{k})$-orbit of a curve in X' and $\pi\varphi^{-1}\colon X' \longrightarrow B$ is a $G(\mathbf{k})$-equivariant rank 2 fibration (see Definition 2.11). We call φ a *link of type I*.

(type II) Either $B = B'$ is a curve or a point, both η and η' are contractions of the $G(\mathbf{k})$-orbit of a curve and $\pi\eta\colon Y \longrightarrow B$ is a $G(\mathbf{k})$-equivariant rank 2 fibration. We call φ a *link of type II*.

(type III) B is a curve, B' is a point, φ is the contraction of the $G(\mathbf{k})$-orbit of a curve and $\pi'\varphi\colon X \longrightarrow B$ is a $G(\mathbf{k})$-equivariant rank 2 fibration. We call φ a *link of type III*. Its inverse is a link of type I.

(type IV) B' and B' are curves, φ is an $G(\mathbf{k})$-equivariant isomorphism not preserving the conic fibrations X/B and X'/B', and $X/*$ is a $G(\mathbf{k})$-equivariant rank 2 fibration. We call φ a *link of type IV*.

For $G = \{1\}$ we recover the classical definition of a Sarkisov link over $\mathbf{k}$.

The statement of Theorem 7.2 for $G = \{1\}$ is [19, Theorem 2.5]. Its proof can be made $G(\mathbf{k})$-equivariant and G-equivariant because for a geometrically rational variety X, the $G_{\overline{\mathbf{k}}} \times \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ has finite action on $\mathrm{NS}(X_{\overline{\mathbf{k}}})$ and $G(\mathbf{k})$ has finite action on $\mathrm{NS}(X)$.

Theorem 7.2 (Equivariant version of [19, Theorem 2.5])**.** *Let G be an affine algebraic group. Any $G(\mathbf{k})$-equivariant birational map between two geometrically rational surfaces that are $G(\mathbf{k})$-Mori fibre spaces is the composition of $G(\mathbf{k})$-equivariant Sarkisov links and isomorphisms.*

The same statement holds if we replace $G(\mathbf{k})$ by G.

To study conjugacy classes of the automorphism groups of the surfaces in Theorem 1.1, it therefore suffices to study equivariant Sarkisov links between them.

Remark 7.3. Definition 7.1 implies the following properties. Let $\phi\colon X/B \dashrightarrow X'/B'$ be an equivariant link.

(1) If ϕ is a link of type I, then B is a point, X/B is an equivariant rank 1 fibration above a point and X'/B is an equivariant rank 2 fibration above a point. Equivariant rank s fibrations above a point are in particular (non-equivariant) rank r fibrations above a point for some $r \geqslant s$, see Definition 2.11, and so they are del Pezzo surfaces, see Definition 2.4. So both X and X' are del Pezzo surfaces. By symmetry, the same holds for a link of type III.

(2) If ϕ is a link of type II and $B = B'$ a point, then X/B and X'/B are equivariant rank 1 fibrations above a point, and Y/B is an equivariant rank 2 fibration above a point. Again, in particular, X, X' and Y are all del Pezzo surfaces.

Many of the surfaces in Theorem 1.1 are equivariant Mori fibre spaces with respect to their automorphism group, as well as to the group of $\mathbf{k}$-points of their automorphism group, and the restrictions for the possible $\mathrm{Aut}_{\mathbf{k}}(X)$-links are also restrictions on the possibilities of $\mathrm{Aut}(X)$-links.

We now classify the $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant links starting from a surface X from Theorem 1.1 in the order (1–3), (5a), (5(b)ii–5(b)iv), (5(b)i), (4) and (6).

7.2. $\mathrm{Aut}_{\mathbf{k}}(X)$**-equivariant links of del Pezzo surfaces of degree** 8 **and** 9**.** We show that there are no $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant links starting from a $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space X that is a rational del Pezzo surface of degree 8 or 9.

Lemma 7.4. *(1) $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ does not have any orbits in $\mathbb{P}^2$ with $d \in \{1, \dots, 8\}$ geometric components that are in general position.*

(2) For $X = \mathbb{F}_0$ and $X = \mathcal{Q}^L$, $\mathrm{Aut}_{\mathbf{k}}(X)$ does not have any orbits in X with $d \in \{1, \dots, 7\}$ geometric components that are in general position.

Proof. (1) Lemma 2.6 implies the claim for $1 \leqslant d \leqslant 4$. If $\mathbf{k}$ is infinite and if $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ had an orbit with $5 \leqslant d \leqslant 8$ geometric components, then $\alpha^{d!} = \mathrm{id}$ for any $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$, which is false. Suppose that $\mathbf{k}$ is finite and let $q := |\mathbf{k}| \geqslant 2$. Let $p = \{p_1, \dots, p_e\}$ be a point in $\mathbb{P}^2$ of degree $e \geqslant 5$ and $L/\mathbf{k}$ be the smallest field extension such that $p_1, \dots, p_e \in \mathbb{P}^2(L)$. We view $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ as an abstract subgroup of $\mathrm{Aut}_L(\mathbb{P}^2)$, which gives us

$$1 = |\cap_{i=1}^e \mathrm{Stab}_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)}(p_i)| = |\mathrm{Stab}_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)}(p_1)| = \frac{|\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)|}{|\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)\text{-orbit of } p_1 \text{ in } \mathbb{P}^2(L)|}.$$

Moreover, we have $|\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)| = q^3(q^3-1)(q^2-1) > q^3 \geqslant 8$, and hence the $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$-orbit of p in $\mathbb{P}^2$ has $\geqslant 9$ geometric components.

(2) For $X = \mathbb{F}_0$ and $d = 1, 2$, the claim follows from Remark 2.7. For $X = \mathcal{Q}^L$, the claim follows from Remark 2.7 for $d = 1$, from Lemma 3.6 for $d = 2$. Let $L/\mathbf{k}$ be a quadratic extension such that $\mathcal{Q}^L_L \simeq \mathbb{P}^1_L \times \mathbb{P}^1_L$, and by Lemma 3.5 we have $\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L) \simeq \mathrm{PGL}_2(L) \rtimes \mathbb{Z}/2$. For $3 \leqslant d \leqslant 7$, we can repeat the argument of (1) for $\mathbb{F}_0$ and $\mathcal{Q}^L$ by using that for a finite field $\mathbf{k}$ with $q := |\mathbf{k}| \geqslant 2$ we have

$$|\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_0)| = 2|\mathrm{PGL}_2(\mathbf{k})|^2 = 2q^2(q^2-1)^2 > 8$$
$$|\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)| = 2|\mathrm{PGL}_2(L)| = 2q^2(q^4-1) > 8.$$

□

Lemma 7.5. *There is no $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant link starting from $X = \mathbb{P}^2$, $X = \mathcal{Q}^L$ or $X = \mathbb{F}_0$.*

Proof. Since $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 1$, the only $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant links starting from X are of type I or II. Moreover, $\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_0)$-equivariant links starting from $\mathbb{F}_0$ can be treated like the ones starting from $\mathcal{Q}^L$ because $\mathrm{NS}(\mathbb{F}_0)^{\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_0)} = \mathbb{Z}(f_1 + f_2) = \mathrm{NS}(\mathcal{Q}^L)$, where f_1, f_2 are the fibres of the two projections of $\mathbb{F}_0$.

By Remark 7.3, an $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$-equivariant link of type I or II starting from $\mathbb{P}^2$ blows up an orbit with $\leqslant 8$ geometric components that are in general position, and by Lemma 7.4(1), there is no such orbit. An $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant link of type I or II starting from $X = \mathcal{Q}^L$ or $X = \mathbb{F}_0$ blows up an orbit with $\leqslant 7$ geometric components that are in general position, and by Lemma 7.4(2), there is no such orbit. □

7.3. $\mathrm{Aut}_{\mathbf{k}}(X)$**-equivariant links of del Pezzo surfaces of degree** 6 **(5a).** These del Pezzo surfaces are Mori fibre spaces. We will show that there are no $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant links starting from X.

Recall from Lemma 4.6 and Lemma 4.7 that there is a quadratic extension $L/\mathbf{k}$ such that X_L is obtained by blowing up a point $p = \{p_1, p_2, p_3\}$ in $\mathbb{P}^2$ of degree 3. We denote by $\pi\colon X_L \longrightarrow \mathbb{P}^2_L$ the blow-up of p. Recall that $\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}$ acts rationally on $\mathbb{P}^2$; its generator ψ_g is not defined at p and sends a general line onto a conic through p. Recall that if X is rational, it has a rational point by Proposition 2.9.

Lemma 7.6. *Let X be a del Pezzo surface of degree 6 from Theorem 1.1(5a) and fix $s \in X(\mathbf{k})$. The map*

$$\mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\langle\psi_g\rangle} \longrightarrow X(\mathbf{k}), \quad \alpha \mapsto \pi^{-1}(\alpha(\pi(s)) = (\pi^{-1}\alpha\pi)(s)$$

is bijective.

Proof. The map is injective, because these automorphisms already fix p_1, p_2, p_3. For any $t \in X(\mathbf{k})$, we have $\pi(t) \in \mathbb{P}^2_L(L)$, and by Lemma 2.6 there exists a unique element of $\alpha_t \in \mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)$ such that $\alpha_t(\pi(s)) = t$. Then $\pi^{-1}\alpha_t\pi \in \mathrm{Aut}_L(X)$ and its conjugate by the generator of $\mathrm{Gal}(L/\mathbf{k})$ is still contained in $\mathrm{Aut}_L(X)$ and preserves each edge of the hexagon, hence $\psi_g\alpha_t\psi_g\alpha_t^{-1} \in \mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)$. The automorphism $\psi_g\alpha_t\psi_g\alpha_t^{-1}$ fixes $p_1, p_2, p_3, \pi(t)$, so it is the identity, and therefore $\alpha_t \in \mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\langle\psi_g\rangle}$. □

Lemma 7.7. *Let X be a del Pezzo surface of degree 6 from Theorem 1.1(5a). Then $|X(\mathbf{k})| \geqslant 7$ if $|\mathbf{k}| \geqslant 3$ and $|X(\mathbf{k})| = 3$ if $|\mathbf{k}| = 2$. Moreover, in the latter case the blow-up of $X(\mathbf{k})$ is a del Pezzo surface.*

Proof. If $\mathbf{k}$ is infinite, then $\mathbb{P}^2(\mathbf{k})$ is dense in $\mathbb{P}^2(\overline{\mathbf{k}})$, and hence $X(\mathbf{k})$ is infinite. If $\mathbf{k}$ is finite, pick a rational point $r \in X(\mathbf{k})$. There exists a link of type II $\phi\colon X \dashrightarrow \mathcal{Q}^L$ that is not defined at r and contracts a curve with three geometric components passing through r, see Figure 2. If $Z \longrightarrow X$ is the blow-up of r and $L/\mathbf{k}$ a quadratic extension such that $\mathcal{Q}_L = \mathbb{P}^1_L \times \mathbb{P}^1_L$, we have

$$q^2 + 1 = |\mathbb{P}^1(L)| = |\mathcal{Q}^L(\mathbf{k})| = |Z(\mathbf{k})| = |X(\mathbf{k})| - 1 + |\mathbb{P}^1(\mathbf{k})| = |X(\mathbf{k})| + q$$

because the exceptional divisor of r is isomorphic to $\mathbb{P}^1_{\mathbf{k}}$. It follows that $|X(\mathbf{k})| = q^2 - q + 1 = q(q-1) + 1$.

Suppose now that $|\mathbf{k}| = 2$ and so $|X(\mathbf{k})| = 3$. Then $X(\mathbf{k})$ is the image of the five points $\mathcal{Q}^L(\mathbf{k})$ by ϕ, and it suffices to show that the blow-up of $\mathcal{Q}^L(\mathbf{k})$ is a del Pezzo surface. We write $L = \mathbf{k}(a)$, where $a^2 + a + 1 = 0$. The set $\mathcal{Q}^L(\mathbf{k})$ consists of

$$([1:0],[1:0]), ([0:1],[0:1]), ([1:1],[1:1]), ([1:a],[1:a^2]), ([1:a^2],[1:a])$$

and we check that they are not contained in any fibre of $\mathcal{Q}^L_L$ nor in any bidegree $(1,1)$-curve. This yields the claim. □

Lemma 7.8. *Let X be a rational del Pezzo surface as in Theorem 1.1(5a).*

(1) If $|\mathbf{k}| \geqslant 3$, then X does not contain any $\mathrm{Aut}_{\mathbf{k}}(X)$-orbits with $\leqslant 5$ geometric components.

(2) If $|\mathbf{k}| = 2$, there is exactly one $\mathrm{Aut}_{\mathbf{k}}(X)$-orbit of X with $\leqslant 5$ geometric components, namely $X(\mathbf{k})$.

Proof. Since $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ acts transitively on the edges of the hexagon, any orbit with $\leqslant 5$ geometric components is outside of it. Let $D \subset \mathbb{P}^2_L$ be the image of the hexagon by π.

Suppose that $|\mathbf{k}| \geqslant 3$. By Lemma 7.7, we have $|X(\mathbf{k})| \geqslant 7$, so Lemma 7.6 implies that the group $\mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\langle\psi_g\rangle}$ has $\geqslant 7$ elements. It acts faithfully on $\mathbb{P}^2\backslash D$, hence any $\mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\langle\psi_g\rangle}$-orbit in $\mathbb{P}^2\backslash D$ has $\geqslant 7$ geometric components. It follows that $\mathrm{Aut}_{\mathbf{k}}(X)$ has no orbits with $\leqslant 5$ geometric components on X.

Suppose now that $|\mathbf{k}| = 2$ and let $L/\mathbf{k}$ be the extension of degree 2. We show that $\pi\,\mathrm{Aut}_{\mathbf{k}}(X)\pi^{-1}$-orbit of any point in $\mathbb{P}^2_L\backslash D$ has either 3 or $\geqslant 6$ elements, and that $\pi(X(\mathbf{k}))$ is the only orbit with 3 elements. Let $\varphi_p \in \mathrm{Bir}_L(\mathbb{P}^2)$ be the quadratic involution from Lemma 4.6(4) and Lemma 4.7(4) that lifts to an automorphism $\tilde{\varphi}_p = \pi^{-1}\varphi_p\pi$ on X over $\mathbf{k}$ inducing a rotation of order 2 on the hexagon of X. By Lemma 4.6(4) (resp. Lemma 4.7(4)) the group

$$\mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\langle\psi_g\rangle} \rtimes \langle\varphi_p\rangle$$

is isomorphic to a subgroup of $\mathrm{Aut}_{\mathbf{k}}(X)$. Lemma 7.7 and Lemma 7.6 imply that $\mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\langle\psi_g\rangle}$ has 3 elements, and it acts faithfully on $\mathbb{P}^2_L\backslash D$. Over $\overline{\mathbf{k}}$, the involution φ_p is conjugate to the involution $[x : y : z] \dashrightarrow [yz : xz : xy]$, which has a unique fixed point in $\mathbb{P}^2_{\overline{\mathbf{k}}}$, namely $[1 : 1 : 1]$, because $|\mathbf{k}| = 2$. Thus φ_p has a unique fixed point $r \in \mathbb{P}^2_L$. Then $\tilde{r} := \pi^{-1}(r)$ is the unique fixed point of $\tilde{\varphi}_p$ on X, and it is $\mathbf{k}$-rational. We have shown that every $\mathrm{Aut}_{\mathbf{k}}(X)$-orbit in $X(L)\backslash X(\mathbf{k})$ has $\geqslant 6$ elements. The set $X(\mathbf{k})$ is an $\mathrm{Aut}_{\mathbf{k}}(X)$-orbit with 3 elements. □

Lemma 7.9. *Let $|\mathbf{k}| = 2$ and let X be a del Pezzo surface from Theorem 1.1(5(a)i). Any $\mathrm{Aut}_{\mathbf{k}}(X)$-invariant link $\varphi\colon X \dashrightarrow Y$ is a link of type II not defined at $X(\mathbf{k})$, and Y is a del Pezzo surface as in Theorem 1.1(5(b)ii).*

Proof. We have $X(\mathbf{k}) = \{r_1, r_2, r_3\}$, see Lemma 7.7, which is an $\mathrm{Aut}_{\mathbf{k}}(X)$-orbit by Lemma 7.8. For a point $s \in S := \{\pi(r_1), \pi(r_2), \pi(r_3), p_1, p_2, p_3\} \subset \mathbb{P}^2_L$, we denote by C_s the strict transform of the conic in $\mathbb{P}^2_L$ passing through the five points in $S\backslash\{s\}$, and let $L_{r_ir_j}$ be the strict transform of the line in $\mathbb{P}^2_L$ through $\pi(r_i), \pi(r_j)$, $i \neq j$. The curves

$$C_p := C_{p_1} \cup C_{p_2} \cup C_{p_3},\ \ D_1 := C_{r_1} \cup L_{r_2r_3},\ \ D_2 := C_{r_2} \cup L_{r_1r_3},\ \ D_3 := C_{r_3} \cup L_{r_1r_2}$$

and $L_i := L_{r_ip_1} \cup L_{r_ip_2} \cup L_{r_ip_3}$, $i = 1, 2, 3$, are irreducible over $\mathbf{k}$. The curve C_p is $\mathrm{Aut}_{\mathbf{k}}(X)$-invariant, while D_1, D_2, D_3 and L_1, L_2, L_3 make up an $\mathrm{Aut}_{\mathbf{k}}(X)$-orbit, see Lemma 4.6(4) for the generators of $\mathrm{Aut}_{\mathbf{k}}(X)$.

Let $\eta\colon Z \longrightarrow X$ be the blow-up of $X(\mathbf{k})$, which is $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant by Lemma 7.8. The surface Z is a del Pezzo surface of degree 3 by Lemma 7.7. There is at most one way to complete η into an $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant link, because Z is an $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant rank 2 fibration, and hence there are at most two extremal $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant contractions from Z. However, any conic fibration $Z \longrightarrow \mathbb{P}^1$ is given by the fibres of the strict transforms of conics through four fixed points in S or the strict transform of lines through one point in $\mathbb{P}^2_L$, but none of them are $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant. So the link φ has to be of type II.

The only $\mathrm{Aut}_{\mathbf{k}}(X) \times \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves on $Z_{\overline{\mathbf{k}}}$ with $\leqslant 6$ geometric components which are pairwise disjoint are the exceptional divisors of η and the strict transform of C_p. The contraction $\eta'\colon Z \longrightarrow Y$ of the latter induces an $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant link $X \dashrightarrow Y$ to a del Pezzo surface Y of degree 6.

Since the strict transforms of C_{p_i} and C_{r_j} on $Z_{\overline{\mathbf{k}}}$ are disjoint for $i, j = 1, 2, 3$, the hexagon of Y consists in the curve $\eta'(D_1) \cup \eta'(D_2) \cup \eta'(D_3)$. Each component $\eta'(D_i)$ of this union is $\mathbf{k}$-rational, so $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ acts as rotation of order 2 on the hexagon of Y, *i.e.* as in Figure 1(4). By Lemma 4.10, Y is described in Theorem 1.1(5(b)ii). □

Proposition 7.10. *Let X be a del Pezzo surface from Theorem 1.1(5a). Then, if $|\mathbf{k}| \geqslant 3$, there are no $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant links starting from X. If $|\mathbf{k}| = 2$, the only $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant link is the one from Lemma 7.9.*

Proof. Since $\mathrm{rk}\,\mathrm{NS}(X) = 1$, only $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant links of type I or II can start from X. By Remark 7.3, they are not defined at an orbit with $\leqslant 5$ geometric components. By Lemma 7.8, such an orbit only exists for surfaces X as in Theorem 1.1(5a) if $|\mathbf{k}| = 2$. The claim now follows from Lemma 7.9. □

7.4. $\mathrm{Aut}_{\mathbf{k}}(X)$**-equivariant links of del Pezzo surfaces of degree** 6 **(5(b)ii)–(5(b)iv).** Any del Pezzo surface X of degree 6 from Theorem 1.1(5(b)ii)–(5(b)iv) is a $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space, and we show that there are no $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant links starting from X.

Lemma 7.11. *Let X be a del Pezzo surface of degree 6 from Theorem 1.1(5(b)ii). Then any $\mathrm{Aut}_{\mathbf{k}}(X)$-orbit on X has at least 6 geometric components.*

Proof. Let $\pi\colon X \longrightarrow \mathbb{F}_0$ be the contraction of a curve in the hexagon onto the point $p = \{(p_1, p_1), (p_2, p_2)\}$ of degree 2 with $p_i = [a_i : 1]$, $i = 1, 2$. Since $\mathrm{Aut}_{\mathbf{k}}(X)$ acts by $\mathrm{Sym}_3 \times \mathbb{Z}/2$ on the hexagon of X, any orbit with $\leqslant 5$ geometric components is outside of the hexagon. Let $D \subset \mathbb{F}_0$ be the image by π of the hexagon, which contains p, and consider the action of $\pi\,\mathrm{Aut}_{\mathbf{k}}(X)\pi^{-1}$ on $\mathbb{F}_0 \backslash D$. The elements of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, p_1, p_2)$ are exactly those of the form

$$[u : v] \mapsto [(b(a_1 + a_2) + c)u - ba_1a_2v : bu + cv], \quad [b : c] \in \mathbb{P}^1(\mathbf{k})$$

and thus

$$|\,\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, p_1, p_2)|^2 = |\mathbb{P}^1(\mathbf{k})|^2 \geqslant 3^2 = 9.$$

Any non-trivial element of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, p_1, p_2)$ has precisely two fixed points in $\mathbb{P}^1$. It follows that the stabiliser in $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, p_1, p_2)^2$ of any point $p_3 \in (\mathbb{F}_0)_{\overline{\mathbf{k}}} \backslash D_{\overline{\mathbf{k}}}$ is trivial and hence

$$|\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, p_1, p_2)^2\text{-orbit of } p_3 \text{ in } (\mathbb{F}_0 \backslash D)_{\overline{\mathbf{k}}}| = |\,\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, p_1, p_2)^2| \geqslant 9.$$

We have shown that $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, p_1, p_2)^2$ has no orbits on $\mathbb{F}_0 \backslash D$ with $\leqslant 5$ geometric components, and hence that $\pi\,\mathrm{Aut}_{\mathbf{k}}(X)\pi^{-1}$ has not orbits on $\mathbb{F}_0 \backslash D$ with $\leqslant 5$ geometric components. □

Remark 7.12. Let $p = \{p_1, p_2, p_3\}$ be a point of degree 3 in $\mathbb{P}^2$. Fix a point $r \in \mathbb{P}^2(\mathbf{k})$. In particular, the point r is not collinear with any two components of p, and so Lemma 2.6 implies that the map $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3) \longrightarrow \mathbb{P}^2(\mathbf{k})$, $\alpha \mapsto \alpha(r)$ is a bijection.

Lemma 7.13. *Let X be a del Pezzo surface of degree 6 from Theorem 1.1(5(b)iii). Then any $\mathrm{Aut}_{\mathbf{k}}(X)$-orbit on X has $\geqslant 6$ geometric components.*

Proof. Since $\mathrm{Aut}_{\mathbf{k}}(X)$ contains an element inducing a rotation of order 6 on the hexagon of X, the hexagon does not contain $\mathrm{Aut}_{\mathbf{k}}(X)$-orbits with $\leqslant 5$ geometric components. Consider the contraction $\pi\colon X \longrightarrow \mathbb{P}^2$ of a curve in the hexagon of X onto the point $p = \{p_1, p_2, p_3\}$ of degree 3, let $D \subset \mathbb{P}^2$ be the image of the hexagon and consider the action of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3) \subset \pi\,\mathrm{Aut}_{\mathbf{k}}(X)\pi^{-1}$ on $\mathbb{P}^2 \backslash D$. Remark 7.12 implies that $|\,\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3)| = |\mathbb{P}^2(\mathbf{k})| \geqslant 7$. The stabiliser of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3)$ of any point in $(\mathbb{P}^2 \backslash D)_{\overline{\mathbf{k}}}$ is trivial, so in particular all the $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3)$-orbits in $\mathbb{P}^2 \backslash D$ have $\geqslant 7$ geometric components. It follows that $\pi\,\mathrm{Aut}_{\mathbf{k}}(X)\pi^{-1}$ has no orbits in $\mathbb{P}^2 \backslash D$ with $\leqslant 5$ geometric components. □

Lemma 7.14. *Let X be a del Pezzo surface of degree 6 from Theorem 1.1(5(b)iv). The blow-up of X in any finite $\mathrm{Aut}_{\mathbf{k}}(X)$-orbit is not a del Pezzo surface.*

Proof. Let $\pi\colon X \longrightarrow \mathbb{P}^2$ be the contraction of a curve C in the hexagon of X onto the point $p = \{p_1, p_2, p_3\}$ of degree 3. By hypothesis, the splitting field $L/\mathbf{k}$ of p satisfies $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathrm{Sym}_3$, so $\mathbf{k}$ is not finite [27, Theorem 6.5]. Remark 7.12 implies that $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3)$ is infinite. Let $D \subset \mathbb{P}^2$ be the image by π of the hexagon and consider the action of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3) \subset \pi \,\mathrm{Aut}_{\mathbf{k}}(X)\pi^{-1}$ on $\mathbb{P}^2\backslash D$. The stabiliser of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3)$ of any point in $(\mathbb{P}^2\backslash D)_{\overline{\mathbf{k}}}$ is trivial, and hence any $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3)$-orbit on $\mathbb{P}^2\backslash D$ has infinitely many geometric components. It follows that any $\mathrm{Aut}_{\mathbf{k}}(X)$-orbit with finitely many geometric components is contained in the hexagon of X, and so its blow-up is not a del Pezzo surface. □

Proposition 7.15. *There is no $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant link starting from a del Pezzo surface X of degree 6 as in Theorem 1.1(5(b)ii) – (5(b)iv).*

Proof. Since $\mathrm{rk}\,\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 1$, the only $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant links starting from X are of type I or II, and by Remark 7.3, they are not defined in an $\mathrm{Aut}_{\mathbf{k}}(X)$-orbit with $\leqslant 5$ geometric components and its blow-up is a del Pezzo surface. If X is as in Theorem 1.1(5(b)ii)–(5(b)iii) no such orbit exists respectively by Lemma 7.11 and Lemma 7.13. If X is as in Theorem 1.1(5(b)iv), then the blow-up of any such orbit is not a del Pezzo surface by Lemma 7.14. □

7.5. $\mathrm{Aut}_{\mathbf{k}}(X)$**-equivariant links of del Pezzo surfaces of degree 6 (5(b)i).** Studying $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant links for such a del Pezzo surface is a bit more involved. We will show that there are equivariant links starting from X only if $|\mathbf{k}| = 2$ and provide examples. Recall Lemma 4.1 for a description of X.

Lemma 7.16. *Fix homogeneous coordinates in $\mathbb{P}^2$ and consider the subgroup $H \subset \mathrm{PGL}_3(\mathbf{k})$ of permutation matrices. If the H-orbit O of a point in $\{xyz \neq 0\} \subset \mathbb{P}^2$ has $\leqslant 5$ geometric components, it is one of the following:*

(1) $O = \{[1:1:1]\}$,
(2) $O = \{[1:a:a^2], [1:a^2:a]\}$ *with* $a^3 = 1$,
(3) $O = \{[1:a:a], [a:a:1], [a:1:a]\}$ *for some* $a \in \mathbf{k}^*$.

Proof. The H-orbit $O_{\overline{\mathbf{k}}}$ of a point $p := [1:a:b] \in \{xyz \neq 0\}_{\overline{\mathbf{k}}}$ is contained in the set

$$\begin{aligned}&\{[1:a:b],[1:b:a],[a:b:1],[b:a:1],[a:1:b],[b:1:a]\}\\ =&\{[1:a:b],[1:b:a],[1:a^{-1}b:a^{-1}],[1:ab^{-1}:b^{-1}],[1:a^{-1}:a^{-1}b],[1:b^{-1}:ab^{-1}]\}\end{aligned}$$

If p is an H-fixed point, we have $O_{\overline{\mathbf{k}}} = O = \{[1:1:1]\}$. We check that if $|O_{\overline{\mathbf{k}}}| = 2$, then we have $O_{\overline{\mathbf{k}}} = \{[1:a:a^2], [1:a^2:a]\}$ with $a^3 = 1$. If $|O_{\overline{\mathbf{k}}}| = 3$, then $O_{\overline{\mathbf{k}}} = \{[1:1:c], [1:c:1], [1:c^{-1}:c^{-1}]\}$ for some $c \in \mathbf{k}^*$. We also check that $4 \leqslant |O_{\overline{\mathbf{k}}}| \leqslant 5$ is not possible. □

Lemma 7.17. *Let X be the del Pezzo surface of degree 6 from Theorem 1.1(5(b)i).*

(1) If $|\mathbf{k}| \geqslant 4$, then X contains no $\mathrm{Aut}_{\mathbf{k}}(X)$-orbits with $\leqslant 5$ geometric components.
(2) If $|\mathbf{k}| = 3$, then $\mathrm{Aut}_{\mathbf{k}}(X)$ has exactly one orbit on X with $\leqslant 5$ geometric components, namely the orbit $\{([1:\pm1:\mp1], [1:\pm1:\mp1])\}$ with 4 elements. Its blow-up is not a del Pezzo surface.
(3) If $|\mathbf{k}| = 2$, then $\mathrm{Aut}_{\mathbf{k}}(X)$ has exactly two orbits on X with $\leqslant 5$ geometric components, namely the fixed point $([1:1:1], [1:1:1])$ and the point $\{([1:\zeta:\zeta^2], [1:\zeta^2:\zeta]), ([1:\zeta^2:\zeta], [1:\zeta:\zeta^2])\}$ of degree 2, where $\zeta \notin \mathbf{k}$, $\zeta^3 = 1$.

Proof. By Lemma 4.1(2), the group $\mathrm{Aut}_{\mathbf{k}}(X)$ acts transitively on the edges of the hexagon, so the hexagon does not contain $\mathrm{Aut}_{\mathbf{k}}(X)$-orbits with $\leqslant 5$ geometric components. We pick three disjoint edges of the hexagon and consider their contraction $\pi\colon X \longrightarrow \mathbb{P}^2$ onto the coordinate points, which maps the hexagon onto the curve $\{xyz = 0\}$. It remains to study the $\pi\operatorname{Aut}_{\mathbf{k}}(X)\pi^{-1}$-action on $\{xyz \neq 0\}$. The stabiliser subgroup of the subgroup $(\mathbf{k}^*)^2 \subset \pi\operatorname{Aut}_{\mathbf{k}}(X)\pi^{-1}$ of diagonal elements of any point in $\{xyz \neq 0\}$ is trivial. It follows that the $(\mathbf{k}^*)^2$-orbit of any point in $\mathbb{P}^2$ has $\geqslant 9$ geometric components if $|\mathbf{k}^*| \geqslant 3$, proving (1).

Let $2 \leqslant |\mathbf{k}| \leqslant 3$ and recall from Lemma 4.1(2) that $\pi\operatorname{Aut}_{\mathbf{k}}(X)\pi^{-1} \simeq (\mathbf{k}^*)^2 \rtimes (H \times \mathbb{Z}/2)$, where $H = \pi\operatorname{Sym}_3\pi^{-1}$ is the group of permutation matrices in $\operatorname{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ and $\mathbb{Z}/2$ is generated by the involution $(x,y) \dashrightarrow (\frac{1}{x}, \frac{1}{y})$.

If a $\pi\operatorname{Aut}_{\mathbf{k}}(X)\pi^{-1}$-orbit in $\{xyz \neq 0\}$ has $\leqslant 5$ geometric components, then this holds in particular for an H-orbit O, which is one of the following by Lemma 7.16

(i) $O = \{[1:1:1]\}$,
(ii) $O = \{[1:a:a^2],[1:a^2:a]\}$ with $a^3 = 1$,
(iii) $O = \{[1:a:a],[1:1:a^{-1}],[1:a^{-1}:1]\}$ for some $a \in \mathbf{k}^*$.

(3) If $|\mathbf{k}| = 2$, then $\pi\operatorname{Aut}_{\mathbf{k}}(X)\pi^{-1} \simeq (H \times \mathbb{Z}/2)$ and the point $[1:1:1]$ is a fixed point and is equal to (iii) and (ii) for $a = 1$. If $a \notin \mathbf{k}$ and $a^3 = 1$, the point $\{[1:a:a^2],[1:a^2:a]\}$ of degree 2 is a $\pi\operatorname{Aut}_{\mathbf{k}}(X)\pi^{-1}$-fixed point.

(2) If $|\mathbf{k}| = 3$, then the $\pi\operatorname{Aut}_{\mathbf{k}}(X)\pi^{-1}$-orbit of $[1:1:1]$ is the set $O = \{[1:\pm1:\pm1]\}$, which has 4 elements. The $\pi\operatorname{Aut}_{\mathbf{k}}(X)\pi^{-1}$-orbit of a point in (ii) or (iii) is either the orbit of $[1:1:1]$ or has $\geqslant 6$ geometric components. The line $\{y = z\} \subset \mathbb{P}^2$ contains $[1:0:0],[1:-1:-1],[1:1:1]$, so the blow-up of X in $\pi^{-1}(O)$ is not a del Pezzo surface. □

Lemma 7.18. *Let $|\mathbf{k}| = 2$ and let X be the del Pezzo surface of degree 6 from Theorem 1.1(5(b)i). The blow-up of X in any $\operatorname{Aut}_{\mathbf{k}}(X)$-orbit with $\leqslant 5$ geometric components does not admit a $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant fibration over $\mathbb{P}^1$.*

Proof. Let $\pi\colon X \longrightarrow \mathbb{P}^2$ be the blow-up of the coordinate points p_1,p_2,p_3. By Lemma 7.17(3), the only $\operatorname{Aut}_{\mathbf{k}}(X)$-orbits on X with $\leqslant 5$ geometric components are a fixed-point $r \in X(\mathbf{k})$ and a point $q \in X$ of degree 2, both not on the hexagon.

Let $Y \longrightarrow X$ be the blow-up of r and let $Y/\mathbb{P}^1$ be a conic fibration. Its fibres are either the strict transform of the lines through one of p_1,p_2,p_3,r, or the strict transform of the conics through p_1,p_2,p_3,r. Since $\operatorname{Aut}_{\mathbf{k}}(X) \simeq \operatorname{Sym}_3 \times \mathbb{Z}/2$ acts transitively on the edges of the hexagon of X by Lemma 4.1 and the quadratic involution in $\pi\operatorname{Aut}_{\mathbf{k}}(X)\pi^{-1}$ sends a general line through r onto a conic through p_1,p_2,p_3,r, it follows that $Y/\mathbb{P}^1$ is not $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant.

Let $Y \longrightarrow X$ be the blow-up of q and $Y/\mathbb{P}^1$ a conic fibration. Its fibres are the strict transforms of the conics through q and two of p_1,p_2,p_3 or of a line through one of p_1,p_2,p_3. Again, as $\operatorname{Aut}_{\mathbf{k}}(X)$ acts transitively on the edges of the hexagon of X, it follows that $Y/\mathbb{P}^1$ is not $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant. □

Example 7.19. Let $\pi\colon X \longrightarrow \mathbb{P}^2$ be the blow-up of the coordinate points p_1,p_2,p_3 of $\mathbb{P}^2$. If $|\mathbf{k}| = 2$, then by Lemma 4.1(2) the group $\pi\operatorname{Aut}_{\mathbf{k}}(X)\pi^{-1} \simeq \operatorname{Sym}_3 \times \mathbb{Z}/2$ is generated by

$$\alpha\colon [x:y:z] \mapsto [x:z:y], \quad \beta\colon [x:y:z] \mapsto [z:y:x], \quad \sigma\colon (x,y) \dashrightarrow (\frac{1}{x}, \frac{1}{y})$$

(1) If $\operatorname{char}(\mathbf{k}) = 2$, the birational map $\psi_1\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_0$

$$\psi_1\colon [x:y:z] \dashrightarrow ([x-z:y-z],[y(x-z):x(y-z)]),$$
$$\psi_1^{-1}\colon ([u_0:u_1],[v_0:v_1]) \dashrightarrow [u_0(u_0+u_1)v_1 : u_1(u_0+u_1)v_0 : u_0u_1(v_0+v_1)]$$

is not defined at $p_1, p_2, p_3, [1:1:1]$ and contracts the $\pi \operatorname{Aut}_{\mathbf{k}}(X)\pi^{-1}$-orbit $\{(y-z)(x-z)(x-y)=0\}$. If $|\mathbf{k}|=2$, it lifts to an $\operatorname{Aut}_{\mathbf{k}}(X)$-birational map

$$\varphi_1 := \psi_1\pi\colon X \dashrightarrow \mathbb{F}_0$$

not defined at $\pi^{-1}([1:1:1])$, because

$$\begin{aligned}
\psi_1\alpha\psi_1^{-1}&\colon ([u_0:u_1],[v_0:v_1]) \mapsto ([u_0+u_1:u_1],[v_0+v_1:v_1]),\\
\psi_1\beta\psi_1^{-1}&\colon ([u_0:u_1],[v_0:v_1]) \mapsto ([u_0:u_0+u_1],[v_0:v_0+v_1]),\\
\psi_1\sigma\psi_1^{-1}&\colon ([u_0:u_1],[v_0:v_1]) \mapsto ([v_0:v_1],[u_0:u_1])
\end{aligned}$$

are automorphisms of $\mathbb{F}_0$. So $\varphi_1\colon X \dashrightarrow \mathbb{F}_0$ is an $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant link of type II.

(2) Let $\operatorname{char}(\mathbf{k})=2$ and $\zeta \in \overline{\mathbf{k}}\backslash\mathbf{k}$, $\zeta^3=1$ and $q := \{[1:\zeta:\zeta^2],[1:\zeta^2:\zeta]\}$. The birational map $\psi_2\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_0$

$$\begin{aligned}
\psi_2\colon [x:y:z] &\mapstochar\dashrightarrow ([xy+xz+yz : y(x+y+z)],[xy+xz+yz : z(x+y+z)],\\
\psi_2^{-1}\colon ([u_0:u_1],[v_0:v_1]) &\mapstochar\dashrightarrow [u_0v_0(u_1v_0+u_0v_1+u_1v_1) : u_1v_0(u_1v_0+u_0v_1+u_0v_0) :\\
&\qquad u_0v_1(u_1v_0+u_0v_1+u_0v_0)]
\end{aligned}$$

is not defined at p_1,p_2,p_3,q and contracts the rational curves $\{(x+y+z)(xy+xz+yz)=0\}$, and the conic $\{y^2+yz+z^2=0\}$ onto $q' := \{([1:\zeta],[1:\zeta^2]),([1:\zeta^2],[1:\zeta])\}$. Let $\eta\colon X' \longrightarrow \mathbb{F}_0$ be the blow-up of q', which is a del Pezzo surface of degree 6 as in Lemma 4.10 (Figure 1(4)). If $|\mathbf{k}|=2$, the contracted curves are $\operatorname{Aut}_{\mathbf{k}}(X)$-invariant and ψ_2 lifts to an $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant birational map

$$\varphi_2 := \eta^{-1}\psi_2\pi\colon X \dashrightarrow X'$$

not defined at $\pi^{-1}(q)$. Consider the conjugates

$$\begin{aligned}
\psi_2\alpha\psi_2^{-1}&\colon ([u_0:u_1],[v_0:v_1]) \mapsto ([v_0:v_1],[u_0:u_1]),\\
\psi_2\beta\psi_2^{-1}&\colon ([u_0:u_1],[v_0:v_1]) \mapstochar\dashrightarrow ([u_0:u_1],[u_0v_0+(u_1v_0+u_0v_1) : u_1v_1+(u_0v_1+u_1v_0)]),\\
\psi_2\sigma\psi_2^{-1}&\colon ([u_0:u_1],[v_0:v_1]) \mapsto ([u_1:u_0],[v_1:v_0]).
\end{aligned}$$

Then $\psi_2\alpha\psi_2^{-1}, \psi_2\sigma\psi_2^{-1} \in \operatorname{Aut}_{\mathbf{k}}(\mathbb{F}_0)$ exchange the geometric components of q' and exchange or preserve the rulings of $\mathbb{F}_0$, hence lift to elements of $\operatorname{Aut}_{\mathbf{k}}(X')$. The birational involution $\psi_2\beta\psi_2^{-1}$ preserves the first ruling of $\mathbb{F}_0$ and exchanges its sections through the components of q', and it contracts the fibre above $\{[1:\zeta],[1:\zeta^2]\}$ onto q', so it lifts to an automorphism of X'. So $\varphi_2\colon X \dashrightarrow X'$ is an $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant link of type II.

Lemma 7.20. *Let $|\mathbf{k}|=2$ and let X be the del Pezzo surface of degree 6 from Theorem 1.1(5(b)i). Any $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant link of type II starting from X is one of the links φ_1,φ_2 in Example 7.19, up to automorphisms of the target surface.*

Proof. Let φ be an $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant link of type II starting from X and let $\eta\colon Y \longrightarrow X$ be the blow-up of its base-locus. Then $Y \longrightarrow *$ is an $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant rank 2 fibration, and by Remark 7.3 the orbit blown-up by η has $\leqslant 5$ components. Since $\operatorname{rk}\operatorname{NS}(Y)^{\operatorname{Aut}_{\mathbf{k}}(X)}=2$, there are exactly two extremal $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant contractions starting from Y, namely the birational morphisms η and η'. It follows that the orbit blown up by η determines φ up to automorphisms of X'. By Lemma 7.17(3), the only $\operatorname{Aut}_{\mathbf{k}}(X)$-orbits on X are $p := ([1:1:1],[1:1:1])$ and $q := \{([1:\zeta:\zeta^2],[1:\zeta^2:\zeta]),([1:\zeta^2:\zeta],[1:\zeta:\zeta^2])\}$,

$\zeta \notin \mathbf{k}$, $\zeta^3 = 1$. The birational maps $\varphi_1 \colon X \dashrightarrow \mathbb{F}_0$ and $\varphi_2 \colon X \dashrightarrow X'$ in Example 7.19 are $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant links of type II with base-points p and q, respectively. □

Proposition 7.21. *Let X be the del Pezzo surface of degree 6 from Theorem 1.1(5(b)i).*

(1) If $|\mathbf{k}| \geqslant 3$, there is no $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant link starting from X.
(2) If $|\mathbf{k}| = 2$, any $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant link starting from X is one of the $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant links of type II in Example 7.19, up to automorphisms of the target surface.

Proof. Since $\operatorname{rk} \operatorname{NS}(X)^{\operatorname{Aut}_{\mathbf{k}}(X)} = 1$, the only $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant links starting from X are of type I or II, and by Remark 7.3, they are not defined in an $\operatorname{Aut}_{\mathbf{k}}(X)$-orbit with $\leqslant 5$ geometric components and the blow-up of this orbit is a del Pezzo surface.

If $|\mathbf{k}| \geqslant 4$, no such orbits exist by Lemma 7.17(1). If $|\mathbf{k}| = 3$, the blow-up of any $\operatorname{Aut}_{\mathbf{k}}(X)$-orbit X with $\leqslant 5$ geometric components is not a del Pezzo surface by Lemma 7.17(2).

If $|\mathbf{k}| = 2$, Lemma 7.18 implies that the blow-up of any $\operatorname{Aut}_{\mathbf{k}}(X)$-orbit on X with $\leqslant 5$ geometric components does not admit an $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant conic fibration. In particular, there is no $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant link of type I starting from X. By Lemma 7.20, any $\operatorname{Aut}_{\mathbf{k}}(X)$-equivariant link of type II starting from X is one of the birational maps in Example 7.19. □

7.6. $\operatorname{Aut}_{\mathbf{k}}(X,\pi)$**-equivariant links of conic fibrations.** We compute all $\operatorname{Aut}_{\mathbf{k}}(X,\pi)$-equivariant links starting from the conic fibrations listed in Theorem 1.1.

Lemma 7.22. *Let $\pi \colon X \longrightarrow \mathbb{P}^1$ be a conic fibration from Theorem 1.1(6a) such that $\mathbf{k}^*/\mu_n(\mathbf{k})$ is trivial. Let $\pi' \colon Y \longrightarrow \mathbb{P}^1$ be a conic fibration such that $\operatorname{Aut}(Y/\pi')$ is infinite. Suppose that there is a $\operatorname{Aut}_{\mathbf{k}}(X,\pi)$-equivariant link $\psi \colon X \dashrightarrow Y$ of type II. Then $Y \simeq X$.*

Proof. The link ψ preserves the set of singular fibres, of which there are at least 4, and it commutes with the $\operatorname{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-action on the set of geometric components of the singular fibres. It follows from Lemma 2.8 that Y is obtained by blowing up a Hirzebruch surface. Since Y is an $\operatorname{Aut}_{\mathbf{k}}(X,\pi)$-Mori fibre space by definition of an equivariant link, the subgroup $\operatorname{Aut}_{\mathbf{k}}(X,\pi) \subseteq \operatorname{Aut}_{\mathbf{k}}(Y,\pi')$ contains an element exchanging the components of a singular geometric fibre. Moreover, since $\operatorname{Aut}(Y/\pi)$ is infinite by hypothesis, Lemma 5.2 implies that there is a birational morphism $\eta' \colon Y \longrightarrow \mathbb{F}_m$ blowing up points $q_1,\dots,q_s \in S_m$ such that $\sum_{i=1}^s \deg(q_i) = 2m$. By Lemma 5.4(2) and since $\mathbf{k}^*/\mu_n(\mathbf{k})$ is trivial, we have $\operatorname{Aut}_{\mathbf{k}}(X/\pi) = \langle\varphi\rangle \simeq \mathbb{Z}/2$ for some involution φ. By Lemma 5.4(3) it has a fixed curve in X, which is the strict transform C of a hyperelliptic curve C' in $\mathbb{F}_n$ (the irreducible double cover of $\mathbb{P}^1$) ramified at $p_1,\dots,p_s$ and disjoint from S_{-n}. It follows that $C' \sim 2S_{-n} + 2nf = 2S_n$ and hence $C^2 = -4n$ since the strict transform of S_n is a $(-n)$-curve on X. An $\operatorname{Aut}_{\mathbf{k}}(X,\pi)$-orbit contains either 1 or 2 points in the same fibre. The base-points of the $\operatorname{Aut}_{\mathbf{k}}(X,\pi)$-equivariant link ψ are therefore necessarily contained in the $\operatorname{Aut}_{\mathbf{k}}(X,\pi)$-fixed curve C. Since C is a double cover of $\mathbb{P}^1$, it follows that $C^2 = \psi(C)^2$. The map $\psi\varphi\psi^{-1} \in \operatorname{Aut}_{\mathbf{k}}(Y/\pi')$ exchanges the components of each singular fibre, so it also exchanges the two special sections of Y. By Lemma 5.4(3) it fixes a curve $D \subset Y$, which satisfies $D^2 = -4m$ with the same argument as above. It follows that $C = \psi^{-1}(D)$, and now $-4n = C^2 = D^2 = -4m$ implies $n = m$. Since ψ induces the identity on $\mathbb{P}^1$, we conclude that $\{q_1,\dots,q_s\} = \{p_1,\dots,p_r\}$. □

Lemma 7.23. *Suppose that $\pi \colon X \longrightarrow \mathbb{P}^1$ is a conic fibration as in Theorem 1.1(4) or (6). Then there are no $\operatorname{Aut}_{\mathbf{k}}(X,\pi)$-equivariant links of type I, III and IV starting from X. Moreover,*

(1) *if* $X = \mathbb{F}_n$, $n \geqslant 2$, *there are no* $\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_n, \pi_n)$*-equivariant links of type II starting from* $\mathbb{F}_n$.
(2) *If* X *is as in Theorem 1.1(6a) and* $\mathbf{k}^*/\mu_n(\mathbf{k})$ *is non-trivial, there are no* $\mathrm{Aut}_{\mathbf{k}}(X, \pi)$*-equivariant links of type II starting from* X.
(3) *If* X *is as in Theorem 1.1(6b), there are no* $\mathrm{Aut}_{\mathbf{k}}(X, \pi)$*-equivariant links of type II starting from* X.

Proof. Since $\mathrm{NS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X,\pi)} \simeq \mathbb{Z}^2$, no $\mathrm{Aut}_{\mathbf{k}}(X, \pi)$-equivariant links of type I can start from X. An $\mathrm{Aut}_{\mathbf{k}}(X, \pi)$-link of type III can only start from a del Pezzo surface (see Remark 7.3), so not from X. Since $\mathrm{Aut}_{\mathbf{k}}(X, \pi) = \mathrm{Aut}_{\mathbf{k}}(X)$, any automorphism of X preserves the conic bundle structure, so there are no $\mathrm{Aut}_{\mathbf{k}}(X, \pi)$-equivariant links of type IV starting from X.

(1) Suppose that there is a $\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_n)$-equivariant link $\psi\colon \mathbb{F}_n \dashrightarrow Y$ of type II, and let $B \subset \mathbb{F}_n$ be the orbit of base-points and $d \geqslant 1$ its number of geometric components. We have $|\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_n/\pi_n)| = |\mathbf{k}^{n+1}| \geqslant 2^3$ by Remark 5.1, so the $\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_n/\pi_n)$-orbit of any point outside the special section has at least two geometric components in the same geometric fibre. If follows that $B \subset S_{-n}$ and hence ψ is a birational map from $\mathbb{F}_n$ to $\mathbb{F}_{n+d}$ and sends S_{-n} onto $S_{-(n+d)}$. Let $P \in \mathbf{k}[z_0, z_1]_d$ be a homogeneous polynomial defining B. Then ψ is of the form

$$\psi\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+d},\ [y_0 : y_1; z_0 : z_1] \mapstochar\dashrightarrow [Q(z_0, z_1)y_0 : R(z_0, z_1)y_0 + P(z_0, z_1)y_1; z_0 : z_1]$$

for some homogeneous $Q, R \in \mathbf{k}[z_0, z_1]$ of degree d. For any $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_n/\pi_n) \simeq \mathbf{k}[z_0, z_1]_n$ we have $\psi\alpha\psi^{-1} \in \mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_{n+d}/\pi_{n+d})$, and we compute that it implies $\lambda := \frac{P}{Q} \in \mathbf{k}^*$ and hence $\lambda\alpha \in \mathbf{k}[z_0, z_1]_{n+d}$ (see Remark 5.1), contradicting $d \geqslant 1$.

(2) If $\pi\colon X \longrightarrow \mathbb{P}^1$ is a conic fibration as in Theorem 1.1(6a) and the torus subgroup $\mathbf{k}^*/\mu_n(\mathbf{k}) \subset \mathrm{Aut}_{\mathbf{k}}(X/\pi)$ is non-trivial, then the $\mathrm{Aut}_{\mathbf{k}}(X/\pi)$-orbit of a point on a smooth fibre outside the two $(-n)$-sections has at least two geometric components in the same smooth fibre. Since $\mathbb{Z}/2 \subset \mathrm{Aut}_{\mathbf{k}}(X/\pi)$ exchanges the two $(-n)$-sections, the same holds for any point contained in them. It follows that there are no $\mathrm{Aut}_{\mathbf{k}}(X, \pi)$-equivariant links of type II starting from X.

(3) Let $\pi\colon X \longrightarrow \mathbb{P}^1$ be a conic fibration as in Theorem 1.1(6b). Consider the subgroup $\mathrm{SO}^{L,L'}(\mathbf{k})$ of $\mathrm{Aut}_{\mathbf{k}}(X/\pi)$ fixing the geometric components of the special double section E from Lemma 5.10(2). Let us show that $|\mathrm{SO}^{L,L'}(\mathbf{k})| \geqslant 2$. From Lemma 4.14 we obtain:

- If L, L' are not $\mathbf{k}$-isomorphic, then $\mathbf{k}$ is infinite, and so $\mathrm{SO}^{L,L'}(\mathbf{k}) \simeq \mathbf{k}^*$ is infinite.
- If $L = L'$, then $\mathrm{SO}^{L,L}(\mathbf{k}) \simeq \{\alpha \in L^* \mid \alpha\alpha^g = 1\}$, where g is the generator of $\mathrm{Gal}(L/\mathbf{k})$. If $|\mathbf{k}| \geqslant 3$, then $\pm 1 \in \mathrm{SO}^{L,L}(\mathbf{k})$, and if $|\mathbf{k}| = 2$, then $|\mathrm{SO}^{L,L}(\mathbf{k})| = |L^*| = 3$.

In any case, it follows that the $\mathrm{Aut}_{\mathbf{k}}(X/\pi)$-orbit of a point on a smooth fibre outside E has at least two geometric components in the same smooth fibre. Since $\mathrm{Aut}_{\mathbf{k}}(X/\pi)$ contains an involution exchanging the geometric components of E by Lemma 5.10(2), the same holds for any point in E. It follows that there are no $\mathrm{Aut}_{\mathbf{k}}(X, \pi)$-equivariant links of type II starting from X. ☐

7.7. **Proof of Theorem 1.2, Corollary 1.3 and Theorem 1.4.** Let G be an affine algebraic group and let X/B be a G-Mori fibre space that is also a $G(\mathbf{k})$-Mori fibre space. A G-equivariant birational map is in particular $G(\mathbf{k})$-equivariant, hence if X is $G(\mathbf{k})$-birationally (super)rigid it is also G-birationally (super)rigid.

On the other hand, G-birationally (super)rigid does not imply $G(\mathbf{k})$-birationally (super)rigid: the next lemma shows that the del Pezzo surface X of degree 6 obtained by blowing up $\mathbb{P}^2$ in three rational points is $\mathrm{Aut}(X)$-birationally superrigid and Example 7.19 shows that X is not even $\mathrm{Aut}_{\mathbf{k}}(X)$-birationally rigid if $|\mathbf{k}| = 2$.

Lemma 7.24. *Any del Pezzo surface X of degree 6 is $\mathrm{Aut}(X)$-birationally superrigid.*

Proof. The surface $X_{\overline{\mathbf{k}}}$ is isomorphic to the del Pezzo surface obtained by blowing up three rational points in $\mathbb{P}^2_{\overline{\mathbf{k}}}$. In particular, $\mathrm{rk}\,\mathrm{NS}(X_{\overline{\mathbf{k}}})^{\mathrm{Aut}_{\overline{\mathbf{k}}}(X)} = 1$ by Lemma 4.1(3), hence X is an $\mathrm{Aut}(X)$-Mori fibre space and there are no $\mathrm{Aut}(X)$-equivariant links of type III or IV starting from X. The base-locus of an $\mathrm{Aut}(X)$-equivariant link of type I or II is an $\mathrm{Aut}_{\overline{\mathbf{k}}}(X) \times \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbit on $X_{\overline{\mathbf{k}}}$, and by Remark 7.3 it has $\leqslant 5$ elements. Lemma 7.17(1) implies that $\mathrm{Aut}_{\overline{\mathbf{k}}}(X) = \mathrm{Aut}(X_{\overline{\mathbf{k}}})$ has no such orbits. By Theorem 7.2, any $\mathrm{Aut}(X)$-equivariant birational map starting from X decomposes into isomorphisms and $\mathrm{Aut}(X)$-equivariant links. As there are no $\mathrm{Aut}(X)$-equivariant links starting from X, it follows that X is $\mathrm{Aut}(X)$-birationally superrigid. □

Proof of Theorem 1.2. (2)–(5) Any surface X as in Theorem 1.1(1)–(3), (5a), and (5b) is a del Pezzo surface that is at the same time a $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space and an $\mathrm{Aut}(X)$-Mori fibre space. Any conic fibration $\pi\colon X \longrightarrow \mathbb{P}^1$ as in Theorem 1.1(4) and (6) has $\mathrm{Aut}(X) = \mathrm{Aut}(X,\pi)$ and $\mathrm{Aut}_{\mathbf{k}}(X) = \mathrm{Aut}_{\mathbf{k}}(X,\pi)$, and it is at the same time a $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space and an $\mathrm{Aut}(X)$-Mori fibre space. By Theorem 7.2, any equivariant birational map between equivariant Mori fibre spaces decomposes into equivariant Sarkisov links, hence in order to show that an equivariant Mori fibre space X/B is equivariantly birationally superrigid, it suffices to show that there are no equivariant links starting from X.

(2) For $X = \mathbb{P}^2$, $X = \mathcal{Q}^L$ and $X = \mathbb{F}_0$ the claim follows from Lemma 7.5 and for $X = \mathbb{F}_n$, $n \geqslant 2$, from Lemma 7.23(1). For X a del Pezzo surface of degree 6 as in (5(b)ii)–(5(b)iv) the claim follows from Proposition 7.15, and for a conic fibration $X/\mathbb{P}^1$ as in (6b) from Lemma 7.23.

(3) For X a del Pezzo surface of degree 6 as in (5a) the claim is Proposition 7.10.

(4) The claim follows from Proposition 7.21.

(5) The claim follows from Lemma 7.22 and Lemma 7.23(2).

(1) It follows from (2)–(5) that for any surface X in Theorem 1.1 there is an algebraic extension $L/\mathbf{k}$ such that X_L is $\mathrm{Aut}_L(X)$-birationally superrigid. Therefore, X is also $\mathrm{Aut}(X)$-birationally superrigid. □

Proof of Corollary 1.3. Theorem 1.1 implies (1). By Theorem 1.2(1), the surfaces X in Theorem 1.1 are $\mathrm{Aut}(X)$-birationally superrigid, so the groups $\mathrm{Aut}(X)$ are maximal and they are conjugate if and only if their surfaces are isomorphic. Theorem 1.1 now implies (2).

By Theorem 1.2(2)–(5), the surfaces X from Theorem 1.1(1)–(4) and (5(b)ii)–(5(b)iv), (6b) are $\mathrm{Aut}_{\mathbf{k}}(X)$-birationally superrigid. The surface X from (6a) are $\mathrm{Aut}_{\mathbf{k}}(X)$-birationally rigid within the set of classes of surfaces from Theorem 1.1. The del Pezzo surfaces X from (5a) and (5(b)i) are $\mathrm{Aut}_{\mathbf{k}}(X)$-birationally superrigid if $|\mathbf{k}| \geqslant 3$. Hence the listed groups $\mathrm{Aut}_{\mathbf{k}}(X)$ are maximal and they are conjugate by a birational map if and only if their surfaces are isomorphic. Theorem 1.1 now implies (3). □

Lemma 7.25. *Let $\mathbf{k}$ be a perfect field and let $F/\mathbf{k}$ be a field extension. The following are equivalent:*

(1) There exists a point p of degree 3 in $\mathbb{P}^2$, not all irreducible components collinear, such that F is the splitting field of p.

(2) F is the splitting field of an irreducible polynomial of degree 3 over $\mathbf{k}$.

(3) *The field extension* $F/\mathbf{k}$ *is Galois and* $\mathrm{Gal}(F/\mathbf{k})$ *is isomorphic to a transitive subgroup of* Sym_3 *(that is to* $\mathbb{Z}/3\mathbb{Z}$ *or* Sym_3*).*

Proof. (1) implies (2): Since the irreducible components p_i of p are not collinear, there is an irreducible conic defined over $\mathbf{k}$ that contains p. With a linear transformation defined over $\mathbf{k}$ this conic can be assumed to be given by $x^2 - yz = 0$, and so $p_i = [a_i : a_i^2 : 1]$ for some $a_i \in F$ for $i = 1, 2, 3$, and $\{a_1, a_2, a_3\}$ is a Galois orbit. Hence $q(t) = (t-a_1)(t-a_2)(t-a_3) \in \mathbf{k}[t]$ is irreducible. The splitting field L of $q(t)$ is $\mathbf{k}(a_1, a_2, a_3) = F$.

(2) implies (1): Similar to above.

(2) implies (3): By assumption F is the splitting field of an irreducible and hence separable polynomial f. Therefore, $F/\mathbf{k}$ is normal and hence Galois. So $\mathrm{Gal}(F/\mathbf{k})$ acts transitively on the three roots of f, hence $\mathrm{Gal}(F/\mathbf{k})$ is isomorphic to a transitive subgroup of Sym_3.

(3) implies (2): Note that by the Primitive element Theorem, there exists $a \in F$ such that $F = \mathbf{k}(a)$. Let f be the minimal polynomial of a over $\mathbf{k}$, hence $\deg(f) = [F : \mathbf{k}] = |\mathrm{Gal}(F/\mathbf{k})| \in \{3, 6\}$. Let L be the splitting field of f, which is a normal extension of $\mathbf{k}$. In particular, $F = \mathbf{k}(a) = L$. Hence, if $\deg(f) = 3$ we are done.

In the other case we have $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3$, so $\deg(f) = 6$. The roots of f form one Galois-orbit. After fixing an isomorphism $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3$, we write $\sigma_{ij} = (ij)$, and we write $\tau = (123)$. So we can write the six roots of f as $a_i = \tau^i(a)$ for $i = 1, 2, 3$ (so $a_3 = a$), and $a_4 = \sigma_{13}(a)$, $a_5 = \sigma_{23}(a)$, $a_6 = \sigma_{12}(a)$. Set

$$b_1 = a_1a_4, b_2 = a_2a_5, b_3 = a_3a_6$$

and note that the σ_{ij} act as transposition of b_i, b_j, and that that τ is the translation $b_1 \mapsto b_2 \mapsto b_3$. So $\{b_1, b_2, b_3\}$ is a $\mathrm{Gal}(F/\mathbf{k})$-orbit of size 3 with minimal polynomial $g = (t - b_1)(t - b_2)(t - b_3) \in \mathbf{k}[t]$. So the splitting field L' of g is contained in F and its Galois group is isomorphic to Sym_3. Hence

$$6 = |\mathrm{Gal}(L'/\mathbf{k})| = [L' : \mathbf{k}] \leqslant [F : \mathbf{k}] = 6,$$

which implies $F = L'$ is the splitting field of an irreducible polynomial of degree 3. □

Proof of Theorem 1.4. By Corollary 1.3(3) it suffices to list the isomorphism classes of the surfaces in Theorem 1.1(1)–(4), (5(b)ii)–(5(b)iv), (6), and for (5a) and (5(b)i) if $|\mathbf{k}| \geqslant 3$.

The plane $\mathbb{P}^2$ is unique up to isomorphism by Châtelet's Theorem, $\mathbb{F}_0$ is unique up to isomorphism by Lemma 3.2(1), and for any $\mathbf{k}$-isomorphism class of quadratic extensions $L/\mathbf{k}$ we have a unique isomorphism class of $\mathcal{Q}^L$, also by Lemma 3.2(1). Hirzebruch surfaces are determined by their special section. The parametrisation of the classes of del Pezzo surfaces from (5a) follows from Lemma 4.6(3), Lemma 4.7(3) and Lemma 7.25. The parametrisation of the classes of del Pezzo surfaces from (5b) follows from Lemma 4.1(1), Lemma 4.2(2), Lemma 4.3(2), Lemma 4.10(2) and Lemma 7.25. The parametrisations for the conic fibrations from (6a) and (6b) follow from Lemma 5.6 and Lemma 5.12. □

8. The image by a quotient homomorphism

We call two Mori fibre spaces $X_1/\mathbb{P}^1$ and $X_2/\mathbb{P}^1$ equivalent if there is a birational map $X_1 \dashrightarrow X_2$ that preserves the fibrations. In particular, if $\varphi\colon X_1 \dashrightarrow X_2$ is a link of type II between Mori fibre spaces $X_1/\mathbb{P}^1$ and $X_2/\mathbb{P}^1$, then these two are equivalent. There is only one class of Mori fibre spaces birational to $\mathbb{F}_1$ [33, Lemma], because all rational points in $\mathbb{P}^2$ are equivalent up to $\mathrm{Aut}(\mathbb{P}^2)$. We denote by J_6 the set of classes of Mori fibre spaces birational to some $\mathcal{S}^{L,L'}$, and by J_5 the set of classes birational to a blow-up of $\mathbb{P}^2$

in a point of degree 4 whose geometric components are in general position. We call two Sarkisov links φ and φ' of type II between conic fibrations equivalent if the conic fibrations are equivalent and and if the base-points of φ and φ' have the same degree. For a class C of equivalent rational Mori fibre spaces, we denote by $M(C)$ the set of equivalence classes of links of type II between conic fibrations in the class C whose base-points have degree $\geqslant 16$.

Proof of Proposition 1.5. First, suppose that $[\bar{\mathbf{k}}:\mathbf{k}]=2$. Then every non-trivial algebraic extension of $\mathbf{k}$ is $\bar{\mathbf{k}}$ by [1, Satz 4] and $\mathbf{k}$ is of characteristic zero [1, p.231]. In particular, $\mathbb{P}^2$ contains no points of degree $\geqslant 3$, and so the only rational Mori fibre spaces are Hirzebruch surfaces and $\mathcal{S}^{\bar{\mathbf{k}},\bar{\mathbf{k}}}\longrightarrow\mathbb{P}^1$. Moreover, $M(\mathbb{F}_1)$ is empty. By [40, Theorem 1.3], there is a surjective homomorphism $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)\longrightarrow\bigoplus_I\mathbb{Z}/2$, where $|I|=|\mathbb{R}|$. In fact, by construction of the homomorphism, there is a natural bijection $I\longrightarrow\{\frac{|a|}{a^2+b^2}\mid a,b\in\mathbb{R},b\neq 0\}$. The whole article [39] can be translated word-by-word over a field $\mathbf{k}$ with $[\bar{\mathbf{k}}:\mathbf{k}]=2$, and consequently we have a surjective homomorphism $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)\longrightarrow\bigoplus_I\mathbb{Z}/2$, where $I=\{\frac{a^2}{a^2+b^2}\mid a,b\in\mathbf{k},b\neq 0\}$ (we replace $|a|$ by a^2), and I has the cardinality of $\mathbf{k}$. If $[\bar{\mathbf{k}}:\mathbf{k}]>2$, the result is [33, Theorem 3, Theorem 4.]. □

Definition 8.1. Let $\mathrm{BirMori}(\mathbb{P}^2)$ be the groupoid of birational maps between Mori fibre spaces birational to $\mathbb{P}^2$. It is generated by Sarkisov links by Theorem 7.2. The homomorphism $\tilde{\Psi}$ of groupoids from [33, Theorem 3, Theorem 4]

$$\begin{array}{ccc}\mathrm{BirMori}(\mathbb{P}'^2) & \xrightarrow{\tilde{\Psi}} & (\bigoplus_{\chi\in M(\mathbb{F}_1)}\mathbb{Z}/2)\ast_{C\in J_5}(\bigoplus_{\chi\in M(C)}\mathbb{Z}/2)\ast(\ast_{C\in J_6}\bigoplus_{\chi\in M(C)}\mathbb{Z}/2)\\ \cup\!| & & \\ \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) & \xrightarrow{\Psi} & \end{array}$$

sends any Sarkisov link of type II between conic fibrations and whose base-point has degree $\geqslant 16$ onto the generator indexed by its class, and it sends all other Sarkisov links and all isomorphisms between Mori fibre spaces to zero.

Remark 8.2. The homomorphism Ψ is non-trivial. Indeed, the surjective homomorphism $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)\longrightarrow(\bigoplus_{I_0}\mathbb{Z}/2)\ast(\ast_{J_5}\bigoplus_I\mathbb{Z}/2)\ast(\ast_{J_6}\bigoplus_I\mathbb{Z}/2)$ from [33, Theorem 4] is obtained by composing Ψ with suitable projections within each abelian factor of the free product, see [33, Proof of Theorem 4 in §6].

We now compute the images by Ψ of $\mathbf{k}$-points of the maximal algebraic subgroups of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ listed in Theorem 1.1.

Remark 8.3. By definition of the groupoid homomorphism $\tilde{\Psi}$ (Definition 8.1), it maps automorphism groups of Mori fibre spaces onto zero, so the groups $\Psi(\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2))$, $\tilde{\Psi}(\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L))$, $\tilde{\Psi}(\mathrm{Aut}_{\mathbf{k}}(\mathbb{F}_n))$, $n\neq 1$, and $\tilde{\Psi}(\mathrm{Aut}(\mathcal{S}^{L,L'},\pi))$ are trivial. A del Pezzo surface X of degree 6 as in Theorem 1.1(5a) is a Mori fibre space by Lemma 4.6 and Lemma 4.7, so $\tilde{\Psi}(\mathrm{Aut}_{\mathbf{k}}(X))$ is trivial as well.

If X is a del Pezzo surface from Theorem 1(5c), there exists a birational morphism $\eta\colon X\longrightarrow\mathcal{Q}^L$ such that $\eta\,\mathrm{Aut}_{\mathbf{k}}(X)\eta^{-1}\subset\mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}^L)$, so in particular $\tilde{\Psi}(\eta\,\mathrm{Aut}_{\mathbf{k}}(X)\eta^{-1})$ is trivial as well.

Lemma 8.4. *Let X be a del Pezzo surface of degree 6 from Theorem 1.1(5b), which is equipped with a birational morphism $\eta\colon X\longrightarrow Y$ to $Y=\mathbb{P}^2$ or $Y=\mathbb{F}_0$. Then $\tilde{\Psi}(\eta\,\mathrm{Aut}_{\mathbf{k}}(X)\eta^{-1})$ is trivial.*

Proof. Let X be a del Pezzo surface of degree 6 from Theorem 1.1(5(b)i), (5(b)iii), and (5(b)iv), which is the blow-up $\eta\colon X \longrightarrow \mathbb{P}^2$ in three rational points or in a point of degree 3. By Lemma 4.1(2), Lemma 4.2(3) and Lemma 4.3(3), the group $\eta \operatorname{Aut}_{\mathbf{k}}(X)\eta^{-1}$ is generated by subgroups of $\operatorname{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ and a quadratic involution of $\mathbb{P}^2$ that has either three rational base-points or is a Sarkisov link of type II with a base-point of degree 3. It follows from the definition of $\tilde{\Psi}$ (Definition 8.1) that $\tilde{\Psi}(\eta \operatorname{Aut}_{\mathbf{k}}(X)\eta^{-1})$ is trivial.

The del Pezzo surface X of degree 6 from Theorem 1.1(5(b)ii) is the blow-up of $\eta\colon X \longrightarrow \mathbb{F}_0$ in a point of degree 2. By Lemma 4.10(3), the group $\eta \operatorname{Aut}_{\mathbf{k}}(X)\eta^{-1}$ is generated by subgroups of $\operatorname{Aut}_{\mathbf{k}}(\mathbb{F}_0)$ and a birational involution of $\mathbb{F}_0$ that is a link of type II of conic fibrations with a base-point of degree 2. Again it follows that $\tilde{\Psi}(\eta \operatorname{Aut}_{\mathbf{k}}(X)\eta^{-1})$ is trivial. $\square$

Lemma 8.5. *Let $n \geqslant 2$ and let $\varphi\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_n$ be the involution from Example 5.3 with base-points $p_1, \dots, p_r \in \mathbb{F}_n$. Then there exist links $\varphi_1, \dots, \varphi_r$ of type II between Hirzebruch surfaces such that φ_i has a base-point of degree $\deg(p_i)$ and $\varphi = \varphi_r \cdots \varphi_1$.*

Proof. Recall from Example 5.3 that $p_1, \dots, p_r$ are contained in the section $S_n \subset \mathbb{F}_n$ and that the homogeneous polynomials $P_i \in \mathbf{k}[z_0, z_1]_{\deg(p_i)}$ define $\pi(p_i)$. The involution φ is given by

$$\varphi\colon (y_1, z_1) \dashrightarrow \left({}^{P(z_1)}\!/\!_{y_1}, z_1\right)$$

We define $d_0 := 0$ and $d_i := \sum_{j=1}^{i} \deg(p_j)$. For $i = 1, \dots, r$, the birational maps

$$\begin{aligned}
&\varphi_i\colon \mathbb{F}_{n-d_{i-1}} \dashrightarrow \mathbb{F}_{n-d_i},\ (y_1, z_1) \mapstochar\dashrightarrow \left({}^{y_1}\!/\!_{P_i(z_1)}, z_1\right) \quad d_i \leqslant n,\\
&\varphi_i\colon \mathbb{F}_{n-d_{i-1}} \dashrightarrow \mathbb{F}_{d_i-n},\ (y_1, z_1) \mapstochar\dashrightarrow \left({}^{P_i(z_1)}\!/\!_{y_1}, z_1\right) \quad d_{i-1} \leqslant n, d_i > n\\
&\varphi_i\colon \mathbb{F}_{d_{i-1}-n} \dashrightarrow \mathbb{F}_{d_i-n},\ (y_1, z_1) \mapstochar\dashrightarrow (P_i(z_1)y_1, z_1), \quad d_{i-1} > n
\end{aligned}$$

are links of type II with a base-point of degree $\deg(p_i)$, and we compute that $\varphi = \varphi_r \cdots \varphi_1$. $\square$

Lemma 8.6. *Let $\pi\colon X \longrightarrow \mathbb{P}^1$ be a conic fibration from Theorem 1.1(6a) and let $\eta\colon X \longrightarrow \mathbb{F}_n$, $n \geqslant 2$, be the birational morphism blowing up $p_1, \dots, p_r$. Let $\varphi\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_n$ be the involution from Example 5.3 and $\varphi = \varphi_r \cdots \varphi_1$ the decomposition into links of type II from Lemma 8.5. Then $\tilde{\Psi}(\eta \operatorname{Aut}_{\mathbf{k}}(X, \pi)\eta^{-1})$ is generated by the element $\tilde{\Psi}(\varphi) = \tilde{\Psi}(\varphi_r) + \cdots + \tilde{\Psi}(\varphi_1)$.*

Proof. Let $\Delta \subset \mathbb{P}^1$ be the image of the singular fibres of X. By Lemma 5.4(1–2), we have

$$\operatorname{Aut}_{\mathbf{k}}(X, \pi) \simeq \operatorname{Aut}_{\mathbf{k}}(X/\pi) \rtimes \operatorname{Aut}_{\mathbf{k}}(\mathbb{P}^1, \Delta) \quad \text{and} \quad \operatorname{Aut}_{\mathbf{k}}(X/\pi) \simeq H \rtimes \langle \eta^{-1}\varphi\eta \rangle$$

where $\eta H \eta^{-1} \subset \operatorname{Aut}_{\mathbf{k}}(\mathbb{F}_n)$. Moreover, any $\alpha \in \operatorname{Aut}_{\mathbf{k}}(\mathbb{P}^1, \Delta)$ lifts to an element $\tilde{\alpha} \in \operatorname{Aut}_{\mathbf{k}}(\mathbb{F}_n, p_1, \dots, p_r)$, which lifts via η to an element of $\operatorname{Aut}_{\mathbf{k}}(X, \pi)$. It follows from the definition of $\tilde{\Psi}$ that $\tilde{\Psi}(\eta \operatorname{Aut}_{\mathbf{k}}(\mathbb{P}^1, \Delta)\eta^{-1})$ and $\tilde{\Psi}(\eta H \eta^{-1})$ are trivial, and that $\tilde{\Psi}(\eta \operatorname{Aut}_{\mathbf{k}}(X, \pi)\eta^{-1})$ is generated by $\tilde{\Psi}(\varphi) = \tilde{\Psi}(\varphi_r) + \cdots + \tilde{\Psi}(\varphi_1)$. $\square$

Lemma 8.7. *Let $\varphi\colon \mathcal{S}^{L,L'} \dashrightarrow \mathcal{S}^{L,L'}$ be the involution from Example 5.9 with base-points $p_1, \dots, p_r \in \mathcal{S}^{L,L'}$. Then there exist links $\varphi_1, \dots, \varphi_r\colon \mathcal{S}^{L,L'} \dashrightarrow \mathcal{S}^{L,L'}$ of type II over $\mathbb{P}^1$ and $\alpha \in \operatorname{Aut}_{\mathbf{k}}(\mathcal{S}^{L,L'}/\pi)$ such that φ_i has base-point p_i and such that $\varphi = \alpha\varphi_r \cdots \varphi_1$.*

Proof. It suffices to construct the φ_i for the involution φ in the case that $L = L'$, since the involution for the other case is obtained by conjugating φ with a suitable element of $\gamma \in \operatorname{PGL}_2(\overline{\mathbf{k}}) \times \operatorname{PGL}_2(\overline{\mathbf{k}})$, see Example 5.9. Let E_1, E_2 be the geometric components of the unique irreducible curve contracted by any birational contraction $\eta\colon \mathcal{S}^{L,L'} \longrightarrow \mathcal{Q}^L$. For $i = 1, \dots, r$, let $T_{i1}, T_{i2} \in L[x, y]$ be the homogeneous polynomials defining the fibres through the geometric components of the p_i contained in E_1, E_2, respectively. Let $P_1 := T_{11} \cdots T_{r1}$ and $P_2 := T_{12} \cdots T_{r2}$. Recall from Example 5.9 that $\psi := \eta\phi\eta^{-1}$ is of the form

$$\psi\colon ([u_0 : u_1], [v_0 : v_1]) \dashrightarrow ([v_0P_1(u_0v_0, u_1v_1) : v_1P_2(u_0v_0, u_1v_1)], [u_0P_2(u_0v_0, u_1v_1) : u_1P_1(u_0v_0, u_1v_1)])$$

For $i = 1, \ldots, r$, define

$$\psi_i\colon ([u_0 : u_1], [v_0 : v_1]) \dashrightarrow ([u_0T_{i2}(u_0v_0, u_1v_1) : u_1T_{i1}(u_0v_0, u_1v_1)], \\ [v_0T_{i1}(u_0v_0, u_1v_1) : v_1T_{i2}(u_0v_0, u_1v_1)])$$

and let

$$\tilde{\alpha}\colon ([u_0 : u_1], [v_0 : v_1]) \dashrightarrow ([v_0 : v_1], [u_0 : u_1]).$$

Then $\alpha\psi_r \cdots \psi_1 = \psi$. We take $\varphi_i := \eta^{-1}\psi_i\eta$ and $\alpha := \eta^{-1}\tilde{\alpha}\eta$. □

Lemma 8.8. *Let $\pi\colon X \longrightarrow \mathbb{P}^1$ be a conic fibration from Theorem 1.1(6b) and let $\eta\colon X \longrightarrow \mathcal{S}^{L,L'}$ be the birational morphism blowing up $p_1, \ldots, p_r$. Let $\varphi\colon \mathcal{S}^{L,L'} \dashrightarrow \mathcal{S}^{L,L'}$ be the involution from Example 5.9 and let $\varphi = \alpha\varphi_r \cdots \varphi_1$ be the decomposition into links φ_i of type II and an automorphism $\alpha \in \mathrm{Aut}_{\mathbf{k}}(\mathcal{S}^{L,L'}, \pi)$ from Lemma 8.7. Then $\tilde{\Psi}(\eta \mathrm{Aut}_{\mathbf{k}}(X, \pi)\eta^{-1})$ is generated by the element $\tilde{\Psi}(\varphi) = \tilde{\Psi}(\varphi_r) + \cdots + \tilde{\Psi}(\varphi_1)$.*

Proof. Let $\Delta \subset \mathbb{P}^1$ be the image of the singular fibres of X. By Proposition 5.10(1–2), we have

$$\mathrm{Aut}_{\mathbf{k}}(X, \pi) \simeq \mathrm{Aut}_{\mathbf{k}}(X/\pi) \rtimes ((D_{\mathbf{k}}^{L,L'} \rtimes \mathbb{Z}/2) \cap \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, \Delta)), \quad \mathrm{Aut}_{\mathbf{k}}(X/\pi) \simeq H \rtimes \langle \eta^{-1}\varphi\eta \rangle$$

where $\eta H \eta^{-1} \subset \mathrm{Aut}_{\mathbf{k}}(\mathcal{S}^{L,L'}/\pi)$. Moreover, any element of $G := D_{\mathbf{k}}^{L,L'} \rtimes \mathbb{Z}/2 \cap \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, \Delta)$ lifts to an element of $\mathrm{Aut}_{\mathbf{k}}(\mathcal{S}^{L,L'}, \pi)$, which lifts via η to an element of $\mathrm{Aut}_{\mathbf{k}}(X, \pi)$. It follows from the definition of $\tilde{\Psi}$, that $\tilde{\Psi}(\eta G \eta^{-1})$, $\tilde{\Psi}(\eta H \eta^{-1})$ and $\tilde{\Psi}(\alpha)$ are trivial, and hence that $\tilde{\Psi}(\eta \mathrm{Aut}_{\mathbf{k}}(X, \pi)\eta^{-1})$ is generated by $\tilde{\Psi}(\varphi) = \tilde{\Psi}(\varphi_r) + \cdots + \tilde{\Psi}(\varphi_1)$. □

Proof of Proposition 1.7. Let G be an infinite algebraic subgroup of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$. By Theorem 1.1, it is conjugate by a birational map to a subgroup of $\mathrm{Aut}(X)$, where X is one of the surfaces listed in Theorem 1.1. We now compute $\Psi(\theta \mathrm{Aut}_{\mathbf{k}}(X)\theta^{-1})$ for some birational map $\theta\colon \mathbb{P}^2 \dashrightarrow X$. For any birational morphism $\eta\colon X \longrightarrow Y$ to a Mori fibre space Y/B, we have

$$\Psi(\theta \mathrm{Aut}_{\mathbf{k}}(X)\theta^{-1}) = \tilde{\Psi}(\theta^{-1}\eta^{-1})\tilde{\Psi}(\eta \mathrm{Aut}_{\mathbf{k}}(X)\eta^{-1})\tilde{\Psi}(\eta\theta).$$

For the surfaces X from Theorem 1.1(1)–(5), there exists such a birational morphism η such that $\tilde{\Psi}(\eta \mathrm{Aut}_{\mathbf{k}}(X)\eta^{-1})$ is trivial by Remark 8.3 and Lemma 8.4, and hence $\Psi(\theta \mathrm{Aut}_{\mathbf{k}}(X)\theta^{-1})$ is trivial. Hence, if $\Psi(G(\overline{\mathbf{k}}))$ is not trivial then X is as in Theorem 1.1(6) and (1) follows.

Let $X/\mathbb{P}^1$ be a conic fibration from Theorem 1.1(6), which is the blow-up $\eta\colon X \longrightarrow Y$ of points $p_1, \ldots, p_r \in Y$ and $Y = \mathbb{F}_n$, $n \geqslant 2$ or $Y = \mathcal{S}^{L,L'}$. By Lemma 8.6 and Lemma 8.8 the image $\tilde{\Psi}(\eta \mathrm{Aut}_{\mathbf{k}}(X)\eta^{-1})$ is generated by the element $\tilde{\Psi}(\varphi_r) + \cdots + \tilde{\Psi}(\varphi_1)$, where φ_i is a link of type II between conic fibrations in the respective class and whose base-point is of degree $\deg(p_i)$. In particular, since each factor of the free product is abelian, it follows that $\Psi(\theta \mathrm{Aut}_{\mathbf{k}}(X)\theta^{-1})$ is generated by $\tilde{\Psi}(\varphi_r) + \cdots + \tilde{\Psi}(\varphi_1)$.

By definition of $\tilde{\Psi}$ the image $\tilde{\Psi}(\varphi_i)$ is non-trivial if and only if $\deg(p_i) \geqslant 16$. Therefore, if $\tilde{\Psi}(\varphi_r) + \cdots + \tilde{\Psi}(\varphi_1)$ is non-trivial, it is the element indexed by the $i_1, \ldots, i_s$ such that $\deg(p_{i_k}) \geqslant 16$ and we infer that $|\{j \in \{1, \ldots, r\} \mid \deg(p_j) = \deg(p_{i_k})\}|$ is odd for $k = 1, \ldots, s$. This proves (2). In particular, $\Psi(G(\mathbf{k})) \simeq \mathbb{Z}/2\mathbb{Z}$. □

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