TOPOLOGICAL SIMPLICITY OF THE CREMONA GROUPS

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Abstract. The Cremona group is topologically simple when endowed with the Zariski or Euclidean topology, in any dimension ≥ 2 and over any infinite field. Two elements are always connected by an affine line, so the group is path-connected.

1. Introduction

Fixing a field k and an integer n, the *Cremona group of rank n over* k can be described algebraically as the group of automorphisms of the k-algebra $\mathrm{Cr}_n(\mathrm{k}) = \mathrm{Aut}_{\mathrm{k}}(\mathrm{k}(x_1, \ldots, x_n))$ or geometrically as the group $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ of birational transformations of $\mathbb{P}^n$ that are defined over the field k.

In an open problem session held at the international congress (see [Mumfo1974]), D. Mumford asked the following: "Let $G = \mathrm{Aut}_{\mathbb{C}}\mathbb{C}(X, Y)$ be the Cremona group [...]. The problem is to topologize G [...] Is G simple?".

As described in [Serre2010] (see section 2.1 below), one can endow the Cremona group with a natural *Zariski topology*, which is induced by *morphisms* $A \to \mathrm{Bir}_{\mathbb{P}^n}$, where A is an algebraic variety (see §2). In [Blanc2010], it is shown that the group $\mathrm{Bir}_{\mathbb{P}^2}(\mathrm{k})$ is topologically simple when endowed with this topology (i.e. it does not contain any non-trivial closed normal strict subgroup), when k is algebraically closed. In this text, we generalise this result and give a simple proof of the following:

Theorem 1. *For each infinite field* k *and each* $n \geq 1$*, the group* $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ *is topologically simple when endowed with the Zariski topology* (*i.e. it does not contain any non-trivial closed normal strict subgroup*).

Remark 1.1. For each field k, the group $\mathrm{Bir}_{\mathbb{P}^2}(\mathrm{k})$ is not simple as an abstract group [CanLam2013, Lonjo2015]. If $\mathrm{k} = \mathbb{R}$, it contains normal subgroups of index 2^m for each $m \geq 1$ [Zimme2015]. For each $n \geq 3$ and each field k, deciding whether the abstract group $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ is simple or not is a still wide open question.

Remark 1.2. If k is a finite field, the Zariski topology on $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ is the discrete topology (see Lemma 2.8), so the topological simplicity is equivalent to the simplicity as an abstract group, and is therefore false for $n = 2$, and open for $n \geq 3$. For $n = 1$, this is true if and only if $\mathrm{k} = \mathbb{F}_{2^a}$, $a \geq 2$ (see Lemma 2.14).

Recall that a local field is a locally compact topological field with respect to a non-discrete valuation. All examples are $\mathbb{R}$, $\mathbb{C}$ and finite extensions of $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$. If k is

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Date: January 14, 2017.

Both authors acknowledge support by the Swiss National Science Foundation Grant "Birational Geometry" PP00P2_153026.

a local field then there exists a natural topology on $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$, which makes it a Hausdorff topological group, and whose restriction on any algebraic subgroup (for instance on $\mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k}) = \mathrm{PGL}_{n+1}(\mathrm{k})$ and $(\mathrm{PGL}_2(\mathrm{k}))^n \subset \mathrm{Aut}_{(\mathbb{P}^1)^n}(\mathrm{k})$) is the Euclidean topology (the classical topology given by distances between matrices) [BlaFur2013, Theorem 3]. This topology was called *Euclidean topology* of $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$. We will show the following analogue of Theorem 1, for this topology:

Theorem 2. *For each local field* k *and each* $n \geq 2$*, the topological group* $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ *is simple when endowed with the Euclidean topology (i.e. it does not contain any non-trivial closed normal strict subgroup).*

Remark 1.3. The result is, of course, false for $n = 1$, since $\mathrm{PSL}_2(\mathbb{R})$ is a non-trivial normal strict subgroup of $\mathrm{PGL}_2(\mathbb{R})$, closed for the Euclidean topology.

In the 1000-th Bourbaki Seminar [Serre2010], J.-P. Serre asked whether the group $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ is connected with respect to the Zariski topology. When k is algebraically closed, a positive answer is given in [Blanc2010, Théorème 5.1]. We generalise this result (and give a simpler proof of it) as follows:

Theorem 3. *For each infinite field* k*, each* $n \geq 2$ *and each* $f, g \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$*, there is a morphism* $\rho\colon \mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}$*, defined over* k*, such that* $\rho(0) = f$ *and* $\rho(1) = g$*. In particular, the group* $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ *is connected with respect to the Zariski topology.*

The second property is also true for $n = 1$*, although the first one is false.*

For each $n \geq 2$*, the groups* $\mathrm{Bir}_{\mathbb{P}^n}(\mathbb{R})$ *and* $\mathrm{Bir}_{\mathbb{P}^n}(\mathbb{C})$ *are path-connected, and thus connected with respect to the Euclidean topology.*

The authors thank the referee for his careful reading and his suggestions for improving the exposition of this article.

2. Preliminaries

2.1. **The families of birational maps and the Zariski topology induced.** In [Demaz1970], M. Demazure introduced the following functor (that he called Psaut, for pseudo-automorphisms, the name he gave to birational transformations):

Definition 2.1. Let $\mathbf{k}$ be an algebraically closed field, X be an irreducible algebraic variety and A a noetherian scheme, both defined over $\mathbf{k}$. We define

$$\begin{aligned}
\mathrm{Bir}_X(A) &= \left\{ \begin{array}{l} A\text{-birational transformations of } A \times X \text{ inducing an} \\ \text{isomorphism } U \to V, \text{ where } U, V \text{ are open subsets} \\ \text{of } A \times X, \text{ whose projections on } A \text{ are surjective} \end{array} \right\}, \\
\mathrm{Aut}_X(A) &= \left\{ \ A\text{-automorphisms of } A \times X \ \right\} = \mathrm{Bir}_X(A) \cap \mathrm{Aut}(A \times X).
\end{aligned}$$

Remark 2.2. When $A = \mathrm{Spec}(\mathbf{k})$, we see that $\mathrm{Bir}_X(A)$ corresponds to the group of birational transformations of X defined over $\mathbf{k}$, which we will denote by $\mathrm{Bir}_X(\mathbf{k})$. Similarly, $\mathrm{Aut}_X(\mathbf{k})$ corresponds to the group of automorphisms of X defined over $\mathbf{k}$.

For each field k over which X is defined, we will similarly denote by $\mathrm{Bir}_X(\mathrm{k})$ and $\mathrm{Aut}_X(\mathrm{k})$ the group of birational transformations and automorphisms of X defined over k.

Definition 2.1 implicitly gives rise to the following notion of families, or morphisms $A \to \mathrm{Bir}_X(\mathbf{k})$ (as in [Serre2010, Blanc2010, BlaFur2013]):

Definition 2.3. Taking A, X as above, an element $f \in \mathrm{Bir}_X(A)$ and a $\mathbf{k}$-point $a \in A(\mathbf{k})$, we obtain an element $f_a \in \mathrm{Bir}_X(\mathbf{k})$ given by $x \dashrightarrow p_2(f(a,x))$, where $p_2 \colon A \times X \to X$ is the second projection.

The map $a \mapsto f_a$ represents a map from A (more precisely from the $A(\mathbf{k})$-points of A) to $\mathrm{Bir}_X(\mathbf{k})$, and will be called a $\mathbf{k}$-*morphism* (or morphism defined over $\mathbf{k}$) from A to Bir_X. If moreover $f \in \mathrm{Aut}_X(A)$, then f also yields a morphism from A to Aut_X.

If $\mathrm{k} \subset \mathbf{k}$ is a subfield over wich X, A and f are defined, we will also say that the $\mathbf{k}$-morphism above is a k-*morphism*.

Remark 2.4.

(1) If X, Y are two irreducible algebraic varieties and $\psi \colon X \dashrightarrow Y$ is a birational map, all of them defined over an algebraically closed field $\mathbf{k}$, the two functors Bir_X and Bir_Y are isomorphic, via ψ. In other words, morphisms $A \to \mathrm{Bir}_X$ corresponds, via ψ, to morphisms $A \to \mathrm{Bir}_Y$. The same holds with Aut_X and Aut_Y, if ψ is an isomorphism. We further get a bijection between k-morphisms to Bir_X and Bir_Y if X, Y and ψ are defined over a subfield $\mathrm{k} \subset \mathbf{k}$.
(2) If X is projective, the connected component Aut_X° of Aut_X is an algebraic group, so there is a natural notion of morphism from A to Aut_X in this case, and this one coincides with the above definition.
(3) Just like with morphisms of algebraic varieties, for any field extension $\mathrm{k} \subset \mathrm{k}'$, any k-morphism $A \to \mathrm{Bir}_X$ is also a k'-morphism, and thus yields a map $A(\mathrm{k}') \to \mathrm{Bir}_X(\mathrm{k}')$.

Even if Bir_X is not representable by an algebraic variety or an ind-algebraic variety in general [BlaFur2013], we can define a topology on the group $\mathrm{Bir}_X(\mathrm{k})$, given by this functor. This topology is called *Zariski topology* by J.-P. Serre in [Serre2010]:

Definition 2.5. Let X be an irreducible algebraic variety defined over a field k. A subset $F \subseteq \mathrm{Bir}_X(\mathrm{k})$ is *closed in the Zariski topology* if for any k-algebraic variety A and any k-morphism $A \to \mathrm{Bir}_X$ the preimage of F in $A(\mathrm{k})$ is closed.

Remark 2.6. In this definition one can of course replace "any algebraic variety A" with "any *irreducible* algebraic variety A".

Endowed with this topology, $\mathrm{Bir}_{\mathbb{P}^n}(\mathbf{k})$ is connected for each $n \geq 1$, and $\mathrm{Bir}_{\mathbb{P}^2}(\mathbf{k})$ is topologically simple for each algebraically closed field $\mathbf{k}$ [Blanc2010].

Let us make the following observation, whose statement and proof is analogue to classical statements for algebraic varieties:

Lemma 2.7. *Let* k *be a field and X a geometrically irreducible algebraic variety defined over* k. *The Zariski topology on* $\mathrm{Bir}_X(\mathrm{k})$ *is finer than the topology on* $\mathrm{Bir}_X(\mathrm{k})$ *induced by the Zariski topology of* $\mathrm{Bir}_X(\mathbf{k})$, *where* $\mathbf{k}$ *is the algebraic closure of* k.

Proof. We show that for each closed subset $F' \subset \mathrm{Bir}_X(\mathbf{k})$, the set $F = F' \cap \mathrm{Bir}_X(\mathrm{k})$ is closed with respect to the Zariski topology.

To do this, we need to show that the preimage of F by any k-morphism $\rho \colon A \to \mathrm{Bir}_X$ is closed. By definition of the Zariski topology of $\mathrm{Bir}_X(\mathbf{k})$, the set $C = \{a \in A(\mathbf{k}) \mid \rho(a) \in F'\}$ is Zariski closed in $A(\mathbf{k})$. The closure R of $C \cap A(\mathrm{k})$ in $A(\mathbf{k})$ is defined over k

[Sprin2009, Lemma 11.2.4]. Since $R(\mathbf{k}) \subset C(\mathbf{k})$, we have $R \cap A(\mathrm{k}) = R(\mathrm{k}) \subset C \cap A(\mathrm{k}) \subset R \cap A(\mathrm{k})$, so $C \cap A(\mathrm{k}) = R(\mathrm{k})$ is closed in $A(\mathrm{k})$.

It remains to observe that the equality $F = F' \cap \mathrm{Bir}_X(\mathrm{k})$ implies that $C \cap A(\mathrm{k}) = \{a \in A(\mathrm{k}) \mid \rho(a) \in F'\} = \{a \in A(\mathrm{k}) \mid \rho(a) \in F\} = \rho^{-1}(F)$. □

Lemma 2.8. *Let* k *be a finite field and* X *be an algebraic variety defined over* k*. The Zariski topology on* $\mathrm{Bir}_X(\mathrm{k})$ *is the discrete topology.*

Proof. Let us show that any subset $F \subset \mathrm{Bir}_X(\mathrm{k})$ is closed. For this, we take a k-algebraic variety A, a k-morphism $\rho\colon A \to \mathrm{Bir}_X$, and observe that $\rho^{-1}(F)$ is finite in A, hence is closed. □

2.2. **The varieties H_d.** The following algebraic varieties are useful to study morphisms to $\mathrm{Bir}_{\mathbb{P}^n}$.

Definition 2.9. [BlaFur2013, Definition 2.3] Let d, n be positive integers.

(1) We define W_d to be the projective space parametrising, for each field k, equivalence classes of non-zero $(n+1)$-uples $(h_0, \dots, h_n)$ of homogeneous polynomials $h_i \in \mathrm{k}[x_0, \dots, x_n]$ of degree d, where $(h_0, \dots, h_n)$ is equivalent to $(\lambda h_0, \dots, \lambda h_n)$ for any $\lambda \in \mathrm{k}^*$. The equivalence class of $(h_0, \dots, h_n)$ will be denoted by $[h_0 : \cdots : h_n]$.
(2) We define $H_d \subseteq W_d$ to be the set of elements $h = [h_0 : \cdots : h_n] \in W_d$ such that the rational map $\psi_h\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given by
$$[x_0 : \cdots : x_n] \dashrightarrow [h_0(x_0, \dots, x_n) : \cdots : h_n(x_0, \dots, x_n)]$$
is birational. We denote by π_d the map $H_d(\mathrm{k}) \to \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ which sends h onto ψ_h.

Proposition 2.10. *Let* d, n *be positive integers.*

(1) *The set* H_d *is locally closed in the projective space* W_d *and thus inherits the structure of an algebraic variety;*
(2) *The map* π_d *corresponds to a morphism* $H_d \to \mathrm{Bir}_{\mathbb{P}^n}$*, defined over any field. For each field* k*, the image of the corresponding map* $H_d(\mathrm{k}) \to \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ *consists of all birational maps of degree* $\leq d$*.*

Proof. Follows from [BlaFur2013, Lemma 2.4]. □

2.3. **The Euclidean topology.** Suppose that k is a local field.

The Euclidean topology of $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ described in [BlaFur2013, Section 5] is defined as follows: on $W_d(\mathrm{k}) \simeq \mathbb{P}^{(n+1)\binom{n+d}{d}-1}(\mathrm{k})$ we put the classical Euclidean topology, on $H_d(\mathrm{k}) \subset W_d(\mathrm{k})$ the induced topology and on $\pi_d(H_d(\mathrm{k})) = \{f \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k}) \mid \deg(f) \leq d\}$ the quotient topology induced by π_d. The Euclidean topology on $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ is then the inductive limit topology induced by the inclusions
$$\{f \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k}) \mid \deg(f) \leq d\} \to \{f \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k}) \mid \deg(f) \leq d+1\}.$$

Lemma 2.11. *Let* k *be a local field, let* A *be an algebraic variety defined over* k*, and let* $\rho\colon A \to \mathrm{Bir}_{\mathbb{P}^n}$ *be a* k*-morphism. Then the map*
$$A(\mathrm{k}) \to \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$$
is continuous for the Euclidean topologies.

Proof. There exists an open affine covering $(A_i)_{i\in I}$ of A, with respect to the Zariski topology, with the following property: for each $i \in I$ there exists an integer d_i and a morphism of algebraic varieties $\rho_i\colon A_i \to H_{d_i}$, such that the restriction of ρ to A_i is $\pi_{d_i} \circ \rho_i$ [BlaFur2013, Lemma 2.6]. It follows from the construction that the A_i and ρ_i can be assumed to be defined over k.

We take a subset $U \subset \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$, open with respect to the Euclidean topology, and want to show that $\rho^{-1}(U) \subset A(\mathrm{k})$ is open with respect to the Euclidean topology. As all $A_i(\mathrm{k})$ are open in $A(\mathrm{k})$, it suffices to show that $\rho^{-1}(U) \cap A_i(\mathrm{k})$ is open in $A_i(\mathrm{k})$ for each i. This follows from the fact that $\rho|_{A_i} = \pi_{d_i} \circ \rho_i$ and that both π_{d_i} and ρ_i are continuous with respect to the Euclidean topology. □

2.4. **The projective linear group.** Note that $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ contains the algebraic group $\mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k}) = \mathrm{PGL}_{n+1}(\mathrm{k})$ and that the restriction of the Zariski topology to this subgroup corresponds to the usual Zariski topology of the algebraic variety $\mathrm{PGL}_{n+1}(\mathrm{k})$, which can be viewed as the open subset of $\mathbb{P}^{(n+1)^2-1}(\mathrm{k})$, more precisely as complement of the hypersurface given by the vanishing of the determinant.

Let us make the following two observations:

Lemma 2.12. *If* k *is an infinite field and* $n \geq 2$, *then* $\mathrm{PSL}_n(\mathrm{k})$ *is dense in* $\mathrm{PGL}_n(\mathrm{k})$ *with respect to the Zariski topology. Moreover, every non-trivial normal subgroup of* $\mathrm{PGL}_n(\mathrm{k})$ *contains* $\mathrm{PSL}_n(\mathrm{k})$. *In particular,* $\mathrm{PGL}_n(\mathrm{k})$ *does not contain any non-trivial normal strict subgroup which is closed with respect to the Zariski topology.*

Proof. (1) Observe that the group homomorphism $\det\colon \mathrm{GL}_n(\mathrm{k}) \to \mathrm{k}^*$ yields a group homomorphism
$$\det\colon \mathrm{PGL}_n(\mathrm{k}) \to (\mathrm{k}^*)/\{f^n \mid f \in \mathrm{k}^*\},$$
whose kernel is the group $\mathrm{PSL}_n(\mathrm{k})$. We consider the morphism
$$\begin{array}{rcl} \rho\colon \mathbb{A}^1(\mathrm{k}) \setminus \{0\} & \to & \mathrm{PGL}_n(\mathrm{k}) \\ t & \mapsto & \begin{pmatrix} t & 0 \\ 0 & I \end{pmatrix} \end{array}$$
where I is the identity matrix of size $(n-1) \times (n-1)$, and observe that $\rho^{-1}(\mathrm{PSL}_n(\mathrm{k}))$ contains $\{t^n \mid t \in \mathbb{A}^1(\mathrm{k})\}$, which is an infinite subset of $\mathbb{A}^1(\mathrm{k})$ and is therefore dense in $\mathbb{A}^1(\mathrm{k})$. The closure of $\mathrm{PSL}_n(\mathrm{k})$ contains thus $\rho(\mathbb{A}^1(\mathrm{k}) \setminus \{0\})$. As every element of $\mathrm{PGL}_n(\mathrm{k})$ is equal to some $\rho(t)$ modulo $\mathrm{PSL}_n(\mathrm{k})$, we obtain that $\mathrm{PSL}_n(\mathrm{k})$ is dense in $\mathrm{PGL}_n(\mathrm{k})$.

(2) Let $N \subset \mathrm{PGL}_n(\mathrm{k})$ be a normal subgroup with $N \neq \{\mathrm{id}\}$, and let $f \in N$ be a non-trivial element. We want to show that N contains $\mathrm{PSL}_n(\mathrm{k})$. Since the center of $\mathrm{PGL}_n(\mathrm{k})$ is trivial, one can replace f with $\alpha f \alpha^{-1} f^{-1}$, where $\alpha \in \mathrm{PGL}_n(\mathrm{k})$ does not commute with f, and assume that $f \in N \cap \mathrm{PSL}_n(\mathrm{k})$. Then, as $\mathrm{PSL}_n(\mathrm{k})$ is a simple group [Dieud1971, Chapitre II, §2], we obtain $\mathrm{PSL}_n(\mathrm{k}) \subset N$.

(1) and (2) imply that $\mathrm{PGL}_n(\mathrm{k})$ does not contain any non-trivial normal strict subgroup which is closed with respect to the Zariski topology. □

Remark 2.13. Lemma 2.12 does not work for the Euclidean topology. For instance, for each $n \geq 1$, the group $\mathrm{PSL}_{2n}(\mathbb{R}) = \{A \in \mathrm{PGL}_{2n}(\mathbb{R}) \mid \det(A) > 0\}$ is a normal strict subgroup of $\mathrm{PGL}_{2n}(\mathbb{R})$ which is closed with respect to the Euclidean topology.

Lemma 2.14. *Let* k *be a finite field. Then*

(1) $\mathrm{PGL}_2(\mathrm{k}) = \mathrm{PSL}_2(\mathrm{k})$ *if and only if* $\mathrm{char}(\mathrm{k}) = 2$,
(2) $\mathrm{PGL}_2(\mathrm{k})$ *is a simple group if and only if* $\mathrm{k} = \mathbb{F}_{2^a}$, $a \geq 2$.

Proof. (1): As explained before, $\mathrm{PSL}_2(\mathrm{k}) = \mathrm{PGL}_2(\mathrm{k})$ if and only if every element of k^* (or equivalently of k) is a square. As k is finite, the group homomorphism

$$\begin{array}{ccc} \mathrm{k}^* & \to & \mathrm{k}^* \\ x & \mapsto & x^2 \end{array}$$

is surjective if and only if it is injective, and this corresponds to ask that the characteristic of k is 2.

(2): If $\mathrm{char}(\mathrm{k}) \neq 2$, then $\mathrm{PSL}_2(\mathrm{k}) \subsetneq \mathrm{PGL}_2(\mathrm{k})$ is a non-trivial normal subgroup.

If $\mathrm{char}(\mathrm{k}) = 2$, then $\mathrm{PGL}_2(\mathrm{k}) = \mathrm{PSL}_2(\mathrm{k})$ is a simple group if and only if $\mathrm{k} \neq \mathbb{F}_2$ ([Dieud1971, Chapitre II, §2]). □

3. Proof of the results

3.1. The construction associated to fixed points. Let us explain the following simple construction that will be often used in the sequel.

Example 3.1. Let $f \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ be an element fixing the point $p = [1 : 0 : \cdots : 0]$ and that induces a local isomorphism at p.

In the chart $x_0 = 1$, we can write f locally as

$$x = (x_1, \ldots, x_n) \dashrightarrow \left(\frac{p_{1,1}(x) + \cdots + p_{1,m}(x)}{1 + q_{1,1}(x) + \cdots + q_{1,m}(x)}, \ldots, \frac{p_{n,1}(x) + \cdots + p_{n,m}(x)}{1 + q_{n,1}(x) + \cdots + q_{n,m}(x)} \right),$$

where the $p_{i,j}, q_{i,j} \in \mathrm{k}[x_1, \ldots, x_n]$ are homogeneous of degree j. For each $t \in \mathrm{k} \setminus \{0\}$, the element

$$\theta_t \colon (x_1, \ldots, x_n) \mapsto (tx_1, \ldots, tx_n)$$

extends to a linear automorphism of $\mathbb{P}^n(\mathrm{k})$ fixing p. Then the map $t \mapsto (\theta_t)^{-1} \circ f \circ \theta_t$ gives rise to a morphism $F \colon \mathbb{A}^1 \setminus \{0\} \to \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ whose image contains only conjugates of f by linear automorphisms.

Writing F locally, we can observe that F extends to a morphism $\mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ such that $F(0)$ is linear. Indeed, $F(t)$ can be written locally as follows:

$$F(t)(x) = F(t)(x_1, \ldots, x_n) = \\ \left(\frac{p_{1,1}(x) + tp_{1,2}(x) + \cdots + t^{m-1}p_{1,m}(x)}{1 + tq_{1,1}(x) + \cdots + t^m q_{1,m}(x)}, \ldots, \frac{p_{n,1}(x) + tp_{n,2}(x) + \cdots + t^{m-1}p_{n,m}(x)}{1 + tq_{n,1}(x) + \cdots + t^m q_{n,m}(x)} \right),$$

and $F(0)$ corresponds to the derivative (linear part) of F at p, which is locally given by

$$(x_1, \ldots, x_n) \mapsto (p_{1,1}(x), \ldots, p_{n,1}(x))$$

and which is an element of $\mathrm{Aut}_{\mathbb{P}^n}(k) \subset \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ since f was chosen to be a local isomorphism at p.

Using the example above, one can construct k-morphisms $\mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}$.

Proposition 3.2. *Let* k *be a field,* $n \geq 1$*, let* $g \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ *and* $p \in \mathbb{P}^n(\mathrm{k})$ *be a point such that* g *fixes* p *and induces a local isomorphism at* p*. Then there exist* k*-morphisms* $\nu \colon \mathbb{A}^1 \setminus \{0\} \to \mathrm{Aut}_{\mathbb{P}^n}$ *and* $\rho \colon \mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}$ *such that the following hold:*

(1) *For each field extension* $\mathrm{k} \subset \mathrm{k}'$ *and each* $t \in \mathbb{A}^1(\mathrm{k}') \setminus \{0\}$*, we have*

$$\rho(t) = \nu(t)^{-1} \circ g \circ \nu(t).$$

Moreover, $\nu(1) = \mathrm{id}$*, so* $\rho(1) = g$*.*

(2) *The element* $\rho(0)$ *belongs to* $\mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$*. It is the identity if and only if the action of* g *on the tangent space* $T_p(\mathbb{P}^n)$ *is trivial.*

Proof. Conjugating by an element of $\mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$, we can assume that $p = [1 : 0 : \cdots : 0]$. We then choose ν to be given by

$$\nu(t)\colon [x_0 : x_1 : \cdots : x_n] \mapsto [x_0 : tx_1 : \cdots : tx_n],$$

and define $\rho\colon \mathbb{A}^1 \setminus \{0\} \to \mathrm{Bir}_{\mathbb{P}^n}$ by $\rho(t) = \nu(t)^{-1} \circ g \circ \nu(t)$. As it was shown in Example 3.1, the k-morphism ρ extends to a k-morphism $\mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}$ such that $\rho(0) \in \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$. Moreover, this element is trivial if and only if the action of g on the tangent space $T_p(\mathbb{P}^n)$ is trivial. □

3.2. **Closed normal subgroups of the Cremona groups.** As a consequence of Proposition 3.2, we obtain the following result:

Proposition 3.3. *Let* k *be an infinite field. Let* n *be a positive integer. Let* $N \subset \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$*. If* N *is closed with respect to the Zariski topology or to the Euclidean topology (if* k *is a local field), then* $N \cap \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$ *is not the trivial group.*

Proof. We can assume that $n \geq 2$, as the result is trivial for $n = 1$ (in which case $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k}) = \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$) . Let us choose a non-trivial element $f \in N$. As f is a birational transformation, it induces an isomorphism $U \to V$, where $U, V \subset \mathbb{P}^n$ are two non-empty open subsets defined over k. Since k is infinite, $U(\mathrm{k})$ and $V(\mathrm{k})$ are not empty, so we can find $p \in U(\mathrm{k})$, and $q = f(p) \in V(\mathrm{k})$. We can moreover choose $p \neq q$, since $\{p \in U \mid f(p) \neq p\}$ is open and non-empty in U. Let us take an element $\alpha \in \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$ that fixes p and q. The element $g = \alpha^{-1} f^{-1} \alpha f \in N$ fixes p and is a local isomorphism at this point. We can choose α such that the derivative $D_p(g)$ of g at this point is not trivial, since

$$D_p(g) = D_p(\alpha^{-1}) \circ D_q(f^{-1}) \circ D_q(\alpha) \circ D_p(f).$$

Indeed, changing coordinates one can assume that $q = [1 : 0 : \cdots : 0]$, $p = [0 : 1 : 0 : \cdots : 0]$ and can for instance choose $\alpha\colon [x_0 : \cdots : x_n] \mapsto [x_0 + \xi x_2 : x_1 : x_2 : \cdots : x_n]$, for some $\xi \in \mathrm{k}$. This choice yields $D_q(\alpha) = \mathrm{id}$ and gives infinitely many possibilities for $D_p(\alpha^{-1})$.

By Proposition 3.2, there exists a k-morphism $\rho\colon \mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}$ such that $\rho(0) \in \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k}) \setminus \{\mathrm{id}\}$ and such that $\rho(t) \in N$ for each $t \in \mathbb{A}^1(\mathrm{k}) \setminus \{0\}$. Since N is closed (with respect to the Zariski or to the Euclidean topology), $\rho^{-1}(N) \subset \mathbb{A}^1(\mathrm{k})$ is closed (with respect to the Zariski or to the Euclidean topology respectively, see Lemma 2.11 in the latter case) and contains $\mathbb{A}^1(\mathrm{k}) \setminus \{0\}$. For the Zariski topology, one uses the fact that k is infinite to get $\rho^{-1}(N) = \mathbb{A}^1(\mathrm{k})$. For the Euclidean topology, one uses the fact that k is non-discrete to get the same result. In each case, we find that $\rho(0) \in N \cap \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$. □

Lemma 3.4. *Let* k *be an infinite field,* $n \geq 2$ *an integer and* $N \subset \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ *be a normal subgroup, with* $N \cap \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k}) \neq \{\mathrm{id}\}$*. Then* $\mathrm{PGL}_{n+1}(\mathrm{k}) = \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k}) \subset N$*.*

Proof. The group $N \cap \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$ is a non-trivial normal subgroup of $\mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k}) = \mathrm{PGL}_{n+1}(\mathrm{k})$, so contains $\mathrm{PSL}_{n+1}(\mathrm{k})$ by Lemma 2.12.

For each $a \in \mathrm{k}^*$, we define $g_a \in N$ and $h \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ by

$$\begin{array}{llll} g_a\colon & [x_0 : \cdots : x_n] & \mapsto & [x_0 : ax_1 : \frac{1}{a}x_2 : x_3 : \cdots : x_n] \\ h\colon & [x_0 : \cdots : x_n] & \dashrightarrow & [x_0 : x_1 : x_2 \cdot \frac{x_1}{x_0} : x_3 : \cdots : x_n]. \end{array}$$

Then, $g_a' = hg_ah^{-1} \in N$ is given by

$$g_a'\colon \quad [x_0 : \cdots : x_n] \mapsto [x_0 : ax_1 : x_2 : x_3 : \cdots : x_n].$$

As every element of $\mathrm{PGL}_n(\mathrm{k})$ is equal to some g_a' modulo $\mathrm{PSL}_{n+1}(\mathrm{k})$, we obtain that $\mathrm{PGL}_{n+1}(\mathrm{k}) \subset N$. □

Proposition 3.5. *Let* k *be an infinite field,* $n \geq 2$ *an integer and consider* $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ *endowed with the Zariski topology or the Euclidean topology (if* k *is a local field). Then the normal subgroup of* $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ *generated by* $\mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$ *is dense in* $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$.

In particular, $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ *does not contain any non-trivial closed normal strict subgroup.*

Proof. (1) Let $f \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$, $f \neq \mathrm{id}$. It induces an isomorphism $U \to V$, where $U, V \subset \mathbb{P}^n$ are two non-empty open subsets, defined over k. Since k is infinite, we can find $p \in U(\mathrm{k})$. There exist $\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$ such that $g := \alpha_1 f \alpha_2$ fixes p, is a local isomorphism at this point and such that $D_p(g)$ is not trivial. By Proposition 3.2, there exist k-morphisms $\nu\colon \mathbb{A}^1 \setminus \{0\} \to \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$ and $\rho_1\colon \mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ such that $\rho_1(t) = \nu(t)^{-1} \circ g^{-1} \circ \nu(t)$ for each $t \in \mathbb{A}^1(\mathrm{k}) \setminus \{0\}$ and $\rho_1(0) \in \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$. We define a k-morphism

$$\rho_2\colon \mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k}), \quad \rho_2(t) = \alpha_1^{-1} \circ g \circ \rho_1(t) \circ \rho_1(0)^{-1} \circ \alpha_2^{-1}.$$

Since $\alpha_1, \alpha_2, \rho_1(0), \nu(t) \in \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$ for all $t \in \mathbb{A}^1 \setminus \{0\}$, the map

$$\rho_2(t) = \alpha_1^{-1} \circ \left(g \circ \nu(t)^{-1} \circ g^{-1}\right) \circ \nu(t) \circ \rho_1(0)^{-1} \circ \alpha_2^{-1}$$

is contained in the normal subgroup of $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ generated by $\mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$, for each $t \in \mathbb{A}^1 \setminus \{0\}$. Therefore, $f = \rho_2(0)$ is contained in the closure of the normal subgroup of $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ generated by $\mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$.

(2) Let $\{\mathrm{id}\} \neq N \subset \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ be a closed normal subgroup (with respect to the Zariski or to the Euclidean topology). It follows from Proposition 3.3 and Lemma 3.4 that $\mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k}) \subset N$. Since N is closed, it contains the closure of the normal subgroup generated by $\mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$, which is equal to $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$. □

Note that Proposition 3.5, together with Lemma 2.12 (for dimension 1 in the case of the Zariski topology), yields Theorems 1 and 2.

3.3. **Connectedness of the Cremona groups.** The group $\mathrm{Bir}_{\mathbb{P}^n}$ is connected with respect to the Zariski topology [Blanc2010]. More precisely, we have the following:

Proposition 3.6. [Blanc2010, Théorème 5.1] *Let* $\mathbf{k}$ *be an algebraically closed field and* $n \geq 1$. *For each* $f, g \in \mathrm{Bir}_{\mathbb{P}^n}(\mathbf{k})$ *there is an open subset* $U \subset \mathbb{A}^1(\mathbf{k})$ *that contains* 0 *and* 1, *and a morphism* $\rho\colon U \to \mathrm{Bir}_{\mathbb{P}^n}(\mathbf{k})$ *such that* $\rho(0) = f$ *and* $\rho(1) = g$.

This corresponds to saying that $\mathrm{Bir}_{\mathbb{P}^n}(\mathbf{k})$ is "rationally connected". We will generalise this for any field k, and provide a morphism from the whole $\mathbb{A}^1$ (Proposition 3.11 below), showing then that $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ is "$\mathbb{A}^1$-uniruled".

Let us recall the following classical fact.

Lemma 3.7. *For each field* k *and each integer $n \geq 2$, there is an integer m and a* k*-morphism $\rho\colon \mathbb{A}^m \to \mathrm{SL}_n$ such that $\rho(\mathbb{A}^m(\mathrm{k})) = \mathrm{SL}_n(\mathrm{k})$.*

Proof. Using Gauss-Jordan elimination, every element of $\mathrm{SL}_n(\mathrm{k})$ is a product of a diagonal matrix and r elementary matrices of the first kind: matrices of the form $I + \lambda e_{i,j}$, $\lambda \in \mathrm{k}$, $i \neq j$, where $(e_{i,j})_{i,j=1,\ldots,n}$ is the canonical basis of the vector space of $n \times n$-matrices. Moreover, the number r can be chosen to be the same for all elements of $\mathrm{SL}_n(\mathrm{k})$. We then observe that

$$\begin{pmatrix} 1 & \lambda - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda^{-1} - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for each $\lambda \in \mathrm{k}^*$. Using finitely many such products, we obtain then all diagonal elements. This gives the existence of $s \in \mathbb{N}$, only dependent on n, such that every element of $\mathrm{SL}_n(\mathrm{k})$ is a product of s elementary matrices of the first kind.

Denoting by $\nu_{i,j}\colon \mathbb{A}^1 \to \mathrm{SL}_n(\mathrm{k})$ the k-morphism sending λ to $I + \lambda e_{i,j}$, this shows that every element of $\mathrm{SL}_n(\mathrm{k})$ is in the image of a product morphism $\mathbb{A}^m \to \mathrm{SL}_n(\mathrm{k})$ of finitely many $\nu_{i,j}$. The number of such maps being finite, we can enlarge m and obtain one morphism for all maps. □

Corollary 3.8. *For each field* k*, each integer $n \geq 2$ and all $f, g \in \mathrm{PSL}_n(\mathrm{k})$, there exists a* k*-morphism $\nu\colon \mathbb{A}^1 \to \mathrm{PSL}_n$ such that $\nu(0) = f$ and $\nu(1) = g$.*

Proof. It suffices to take a morphism $\rho\colon \mathbb{A}^m \to \mathrm{SL}_n$ as in Lemma 3.7, to choose $v, w \in \mathbb{A}^m(\mathrm{k})$ such that $\rho(v) = f$, $\rho(w) = g$ in $\mathrm{PSL}_n(\mathrm{k})$, and to define $\nu(t) = \rho(v + t(w - v))$. □

Remark 3.9. By construction, Corollary 3.8 also works for $\mathrm{SL}_n(\mathrm{k})$, but is in fact false for $\mathrm{GL}_n(\mathrm{k})$. Indeed, every k-morphism $\nu\colon \mathbb{A}^1 \to \mathrm{GL}_n$ gives rise to a morphism $\det \circ \nu\colon \mathbb{A}^1 \to \mathbb{A}^1 \setminus \{0\}$, which is necessarily constant. As every morphism $\mathbb{A}^1 \to \mathrm{PGL}_n$ lifts to a morphism $\mathbb{A}^1 \to \mathrm{GL}_n$, the same holds for PGL_n.

Example 3.10. Let k be a field, $n \geq 2$ and $\lambda \in \mathrm{k}^*$. We consider $g \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ given by

$$g\colon [x_0 : \cdots : x_n] \mapsto \left[\frac{x_0(x_1 + \lambda x_2) + x_1 x_2}{x_1 + x_2} : x_1 : \cdots : x_n\right]$$

We observe that $p_1 = [0 : 1 : 0 : \cdots : 0]$ and $p_2 = [0 : 0 : 1 : 0 : \cdots : 0]$ are both fixed by g. In local charts $x_1 = 1$ and $x_2 = 1$, the map g becomes:

$$\begin{array}{rcl} [x_0 : 1 : x_2 : x_3 : \cdots : x_n] & \mapsto & \left[\dfrac{x_0(1 + \lambda x_2) + x_2}{x_2 + 1} : 1 : x_2 : x_3 : \cdots : x_n\right] \\ [x_0 : x_1 : 1 : x_3 : \cdots : x_n] & \mapsto & \left[\dfrac{x_0(x_1 + \lambda) + x_1}{x_1 + 1} : x_1 : 1 : x_3 : \cdots : x_n\right] \end{array}$$

Applying Proposition 3.2 to the two fixed points, we get two k-morphisms $\rho_1, \rho_2\colon \mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}$ such that $\rho_1(1) = g = \rho_2(1)$ and $\rho_1(0), \rho_2(0) \in \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$. The two elements are provided by the construction Example 3.1. Choosing for this one the affine coordinates $x_1 \neq 0$ and $x_2 \neq 0$ using permutations of the coordinates, we obtain the following maps corresponding to the linear parts in these affine spaces:

$$\begin{array}{rrcl} \rho_1(0)\colon & [x_0 : x_1 : x_2 : x_3 : \cdots : x_n] & \mapsto & [x_0 + x_2 : x_1 : x_2 : x_3 : \cdots : x_n], \\ \rho_2(0)\colon & [x_0 : x_1 : x_2 : x_3 : \cdots : x_n] & \mapsto & [x_0\lambda + x_1 : x_1 : x_2 : x_3 : \cdots : x_n]. \end{array}$$

We can now give the following generalisation of [Blanc2010, Théorème 5.1] (Proposition 3.6):

Proposition 3.11. *For each infinite field* k*, each integer* $n \geq 2$ *and all* $f, g \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$*, there exists a* k*-morphism* $\nu\colon \mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}$ *such that* $\nu(0) = f$ *and* $\nu(1) = g$*.*

Proof. Multiplying the morphism with f^{-1}, we can assume that $f = \mathrm{id}$. We denote by $N \subset \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ the subset given by

$$N = \left\{ g \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k}) \,\middle|\, \begin{array}{l} \text{there exists a k-morphism } \nu\colon \mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n} \\ \text{such that } \nu(0) = \mathrm{id} \text{ and } \nu(1) = g \end{array} \right\}.$$

If $f, g \in N$ are associated to k-morphisms ν_f, ν_g, we define a k-morphism $\nu_{fg}\colon \mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}$ by $\nu_{fg}(t) = \nu_f(t)\nu_g(t)$, which satisfies $\nu_{fg}(0) = \mathrm{id}$ and $\nu_{fg}(1) = fg$. For each $h \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$, we can also define a morphism $t \mapsto h\nu_f(t)h^{-1}$. Thus, N is a normal subgroup of $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ and it contains $\mathrm{PSL}_{n+1}(\mathrm{k})$ by Corollary 3.8. As N is a priori not closed, we cannot apply Theorem 1. However, we will apply Proposition 3.2 and Example 3.10 to obtain the result.

First, taking $\lambda, g, \rho_1, \rho_2$ as in Example 3.10, the morphisms $t \mapsto \rho_i(t) \circ \rho_i(0)^{-1}$, $i = 1, 2$, show that $g \circ (\rho_1(0))^{-1}, g \circ (\rho_2(0))^{-1} \in N$, which implies that $\rho_1(0) \circ (\rho_2(0))^{-1} \in N$. Since $\rho_1(0) \in \mathrm{PSL}_{n+1}(\mathrm{k}) \subset N$, this implies that

$$\rho_2(0)\colon [x_0 : x_1 : x_2 : x_3 : \cdots : x_n] \mapsto [x_0\lambda + x_1 : x_1 : x_2 : x_3 : \cdots : x_n]$$

belongs to N, for each $\lambda \in \mathrm{k}^*$. Hence, $\mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k}) = \mathrm{PGL}_{n+1}(\mathrm{k}) \subset N$.

Second, we take any $g \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ of degree $d \geq 2$, take a point $p \in \mathbb{P}^n(\mathrm{k})$ such that g induces a local isomorphism at p, choose $\alpha \in \mathrm{PSL}_{n+1}(\mathrm{k})$ such that $\alpha \circ g$ fixes p. Proposition 3.2 yields the existence of a k-morphism $\rho\colon \mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}$ with $\rho(1) = \alpha \circ g$ and $\rho(0) \in \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k})$. Choosing $\rho'\colon \mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}$ given by $\rho'(t) = \rho(t) \circ \rho(0)^{-1}$, we obtain that $\rho'(1) = \alpha \circ g \circ \rho(0)^{-1} \in N$. Since $\alpha, \rho(0) \in \mathrm{Aut}_{\mathbb{P}^n}(\mathrm{k}) \subset N$, this shows that $g \in N$ and concludes the proof. $\square$

Corollary 3.12. *For each infinite field* k *and each* $n \geq 1$*, the group* $\mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ *is connected with respect to the Zariski topology.*

Proof. For $n = 1$, the result follows from the fact that $\mathrm{Bir}_{\mathbb{P}^1} = \mathrm{Aut}_{\mathbb{P}^1} = \mathrm{PGL}_2$ is an open subvariety of $\mathbb{P}^3$. For $n \geq 2$, this follows from Proposition 3.11. $\square$

Corollary 3.13. *For each* $n \geq 2$*, the groups* $\mathrm{Bir}_{\mathbb{P}^n}(\mathbb{R})$ *and* $\mathrm{Bir}_{\mathbb{P}^n}(\mathbb{C})$ *are path-connected, and thus connected with respect to the Euclidean topology.*

Proof. Let us fix $\mathrm{k} = \mathbb{R}$ or $\mathrm{k} = \mathbb{C}$. For each $f, g \in \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ there is a k-morphism $\nu\colon \mathbb{A}^1 \to \mathrm{Bir}_{\mathbb{P}^n}$ such that $\nu(0) = f$ and $\nu(1) = g$ (Proposition 3.11). The corresponding map $\mathrm{k} = \mathbb{A}^1(\mathrm{k}) \to \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$ is continuous with respect to the Euclidean topologies (Lemma 2.11). The restriction of this map to the interval $[0, 1] \subset \mathbb{R} \subset \mathbb{C}$ yields a map $[0, 1] \to \mathrm{Bir}_{\mathbb{P}^n}(\mathrm{k})$, continuous with respect to the Euclidean topologies and sending 0 to f and 1 to g. $\square$

Theorem 3 is now proven, as a consequence of Proposition 3.11 and Corollaries 3.12 and 3.13.

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