FINDING NORMAL SUBGROUPS OF THE CREMONA GROUP

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Abstract. This is a survey on what is known, up to date, on normal subgroups of Cremona groups. There are several different approaches to showing they exist and we will take a look at each of them, more or less in chronological order.

Contents

1. Introduction

Let $\mathbf{k}$ be a field. The Cremona group $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ is the group of birational self-maps of $\mathbb{P}^n_{\mathbf{k}}$, that is, the group of isomorphism between dense open sets of $\mathbb{P}^n_{\mathbf{k}}$. If we choose affine coordinates on $\mathbb{P}^n_{\mathbf{k}}$, they have the form

$$(x_1 \dots, x_n) \dashrightarrow (f_1(x_0, \dots, x_n), \dots, f_n(x_0, \dots, x_n))$$

where $f_1, \dots, f_n \in \mathbf{k}(x_0, \dots, x_n)$ are non-zero. They are well studied if $n = 2$ and harder to study when $n \geqslant 3$. The reader may find in [7, 27, 38] very well written and accessible introductions to plane Cremona groups.

Recall that a group is simple if its only normal subgroups are the trivial group and the whole group itself. If $n = 1$, we have $\mathrm{Bir}(\mathbb{P}^1_{\mathbf{k}}) = \mathrm{Aut}(\mathbb{P}^1_{\mathbf{k}}) \simeq \mathrm{PGL}_2(\mathbf{k})$, which is a simple group when $\mathbf{k}$ is algebraically closed. The earliest reference mentioning the problem of (non-)simplicity of Cremona groups is in a book by F. Enriques in 1895:

> *Tuttavia altre questioni d'indole gruppale relative al gruppo Cremona nel piano* (*ed a più forte ragione in* S_n $n > 2$) *rimangono ancora insolute; ad esempio l'importante questione se il gruppo Cremona contenga alcun sottogruppo invariante* (*questione alla quale sembra probabile si debba rispondere negativamente*).
> [18, p. 116][1]

The problem was also mentioned in the article of V. Iskovskikh in the Encyclopedia:

> *It is not known to date* (1987) *whether the Cremona group is simple.* [23]

The idea of F. Enriques that the Cremona group should be simple was perhaps motivated by the analogy with automorphism groups of projective varieties, such as $\mathrm{Aut}(\mathbb{P}^n) =$

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[1] "However, other group-theoretic questions related to the Cremona group of the plane (and, even more so, of $\mathbb{P}^n$, $n > 2$) remain unsolved; for example, the important question of whether the Cremona group contains any normal subgroup (a question which seems likely to be answered negatively)."

$\mathrm{PGL}_{n+1}(\mathbf{k})$ when $\mathbf{k}$ is algebraically closed. Or perhaps by the fact that the normal subgroup generated by a nice birational map of $\mathbb{P}^2_{\mathbf{k}}$ is the whole of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$:

Lemma 1.1. *Let* $\mathbf{k}$ *be an algebraically closed field and let* $\alpha \in \mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ *be a non-trivial automorphism of* $\mathbb{P}^2_{\mathbf{k}}$*. Then the normal subgroup* $\langle\langle\alpha\rangle\rangle$ *of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ *generated by* α *is equal to* $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$*.*

Proof. Since $\mathbf{k}$ is algebraically closed, $\mathrm{PGL}_3(\mathbf{k}) = \mathrm{PSL}_3(\mathbf{k})$ is a simple group and so we have $\mathrm{PGL}_3(\mathbf{k}) \subset \langle\langle\alpha\rangle\rangle$ and hence $\langle\langle\mathrm{PGL}_3(\mathbf{k})\rangle\rangle \subset \langle\langle\alpha\rangle\rangle$. The Noether-Castelnuovo theorem [12] states that $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is generated by $\mathrm{PGL}_3(\mathbf{k})$ and $\sigma\colon (x,y) \dashrightarrow (\frac{1}{x}, \frac{1}{y})$. Set $h\colon (x,y) \mapsto (1-x, 1-y)$ and compute that $(\sigma h)^3 = \mathrm{id}$. Then $(h\sigma h)\sigma(h\sigma h)^{-1} = (h\sigma h)\sigma(h\sigma h) = h \in \mathrm{PGL}_3(\mathbf{k})$, so σ is contained in $\langle\langle\mathrm{PGL}_3(\mathbf{k})\rangle\rangle$. It follows that $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) = \langle\langle\alpha\rangle\rangle$. □

Lemma 1.2 ([20, Lemma 2])**.** *Let* $\mathbf{k}$ *be an algebraically closed field. If* $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ *preserves a pencil of lines, then the normal subgroup* $\langle\langle f\rangle\rangle$ *of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ *generated by* f *is equal to* $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$*.*

We see from the above lemmas that it is not straight forward at all to find proper normal subgroups of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$, even with amazing calculation skills and/or computing power. In Section 2 we will follow the approach of S. Cantat, S. Lamy and A. Lonjou to show the existence of non-trivial proper normal subgroups of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ over any field [11, 33] and visit the work of S. Cantat, V. Guirardel and A. Lonjou who achieved an almost complete description of infinite order elements of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ generating a proper normal subgroup if $\mathbf{k}$ is algebraically closed [9]. A classification can also be found in [37]. The normal subgroups N found by A. Lonjou are large and moreover their quotients $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})/N$ are SQ-universal, that is, any finitely generated groups embeds into a quotient of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})/N$ [14]. We will also go into the construction of normal subgroups of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ by J. Schneider [36] over perfect fields.

The technique used in [11, 33] cannot be easily generalised to dimension $n \geqslant 3$. In Section 3 we will follow the approach of J. Blanc, S. Lamy and the author who show that $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$, $n \geqslant 3$, is not simple if $\mathbf{k}$ is of characteristic zero by constructing many non-trivial surjective morphisms of groups $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}) \longrightarrow \mathbb{Z}/2$.

We will then look at the approach of H.-Y. Lin and E. Shinder who construct a surjective morphism of groups $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}) \longrightarrow \mathbb{Z}$ for $n \geqslant 3$ and a large family of fields [31]. Last but not least, we will explain the approach of J. Blanc, J. Schneider and E. Yasinsky who construct surjective morphisms of groups $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}) \longrightarrow \mathcal{F}(\mathbb{C})$ to the free group $\mathcal{F}(\mathbb{C})$ over $\mathbb{C}$ for $n \geqslant 4$ and fields of characteristic zero.

The works [1] and [31] leave open (for the moment) the following problems in dimension $n \geqslant 3$:

Problem 1.3. *Let* $\mathbf{k}$ *be a field of characteristic zero. Is* $\mathrm{Bir}(\mathbb{P}^3_{\mathbf{k}})$ *non-simple if* $\mathbf{k}$ *is not a function field over a number field or a function field over an algebraically closed field?*

Problem 1.4. *Let* $\mathbf{k}$ *be a field of positive characteristic. Is* $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ *non-simple if*

- $n = 3$ *and* $\mathbf{k}$ *is not a function field over a finite field?*
- $n = 4$*?*
- $n \geqslant 5$ *and* $\mathbf{k}$ *is finite?*

Hopefully, the reader will find it natural to formulate open problems concerning normal subgroups of Cremona groups or quotient maps of Cremona groups after enjoying this survey.

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2. Normal subgroups in dimension 2

2.1. **The existence of normal subgroups of the plane Cremona group.** In this section, $\mathbf{k}$ is any field, and all surfaces, curves and morphisms are defined over $\mathbf{k}$, and we write $\mathbb{P}^2$ instead of $\mathbb{P}^2_{\mathbf{k}}$. As explained in the introduction (Lemma 1.1 and Lemma 1.2), it is hopeless to just find proper normal subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ by trying to make an educated guess. A different approach is needed and S. Cantat, S. Lamy and A. Lonjou chose to approach the problem with geometric group theory [11, 34].

In a nut shell, their approach is the following: find a hyperbolic space on which $\mathrm{Bir}(\mathbb{P}^2)$ acts by isometries and use geometric group theory to show the existence of normal subgroups of $\mathrm{Bir}(\mathbb{P}^2)$.

2.1.1. *The infinite-dimensional hyperbolic space on which* $\mathrm{Bir}(\mathbb{P}^2)$ *acts.* In a slightly bigger nutshell, the approach is the follwing: the Cremona group does not act on the space of curves $\mathbb{P}^2$ because birational maps that are not automorphisms contract some curves onto points. Let us blow up all points on $\mathbb{P}^2$ and look at the space $Z_C(\mathbb{P}^2)$ of all curves on all these blow-ups. It is the inductive limit of the Néron-Severi spaces of surfaces obtained by blowing up $\mathbb{P}^2$ and it is a $\mathbb{Z}$-module. The intersection form of curves on $\mathbb{P}^2$ lifts to $Z_C(\mathbb{P}^2)$, but the space is "too discrete". If we tensor with $\mathbb{R}$, the space is not complete and we consider its ℓ^2-completion and obtain a Hilbert space $\mathcal{Z}(\mathbb{P}^2)$, called *Picard-Manin space* of $\mathbb{P}^2$. The intersection form on $\mathcal{Z}(\mathbb{P}^2)$ cuts out from $\mathcal{Z}(\mathbb{P}^2)$ a hyperbolic space $\mathbb{H}^\infty(\mathbb{P}^2)$ of infinite dimension, and $\mathrm{Bir}(\mathbb{P}^2)$ acts on $\mathbb{H}^\infty(\mathbb{P}^2)$ by isometries.

Let us now be more precise about the construction of $\mathbb{H}^\infty(\mathbb{P}^2)$. The pullback of a birational morphism $\pi\colon S \longrightarrow \mathbb{P}^2_{\mathbf{k}}$ induces an injection $NS(\mathbb{P}^2) \hookrightarrow NS(S)$, $D \mapsto \pi^*D$. We consider the inductive limit

$$Z_C(\mathbb{P}^2) := \lim_{S\longrightarrow\mathbb{P}^2} NS(S)$$

over all birational morphisms $S \longrightarrow \mathbb{P}^2$. The C in $Z_C(S)$ is a notation referring to Cartier b-divisors, see [19] for more information. The inductive limit identifies divisors as follows: if $S_2 \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} \mathbb{P}^2$ are two birational morphisms and D a curve on $\mathbb{P}^2$, then the class $\bar{D} \in \mathcal{Z}(\mathbb{P}^2)$ of D represents D and π_1^*D and $(\pi_1\circ\pi_2)^*D$. We can think

$$Z_C(\mathbb{P}^2) = NS(\mathbb{P}^2) \oplus \left(\oplus_{p\in\mathcal{B}(\mathbb{P}^2)}\mathbb{Z}e_p\right),$$

where e_p is the class of the exceptional divisors of p and where $\mathcal{B}(\mathbb{P}^2)$ is the set of all points of all smooth projective surfaces with a birational morphism to $\mathbb{P}^2$ and we identify points around which the birational morphism is an isomorphism (it is called *Bubble space of* $\mathbb{P}^2$).

The set $Z_C(\mathbb{P}^2)$ is endowed with a natural intersection form of signature $(1,\infty)$: if C, D are curves on $\mathbb{P}^2$, then $(\pi_1\circ\pi_2)^*C\cdot(\pi_1\circ\pi_2)^*D = \pi_1^*C\cdot\pi_1^*D = C\cdot D$. If C, D are two curves on surfaces S_1, S_2, respectively, with a birational morphism to $\mathbb{P}^2$, consider the resolution $(p,q)\colon S_0 \longrightarrow S_1\times S_2$ of the induced birational map $S_1 \dashrightarrow S_2$. We can intersect p^*C and q^*D, and the above shows that this intersection does not depend on the choice of resolution. So, if $c, d\in Z_C(\mathbb{P}^2)$ are the classes of the curves C, D, the integer $c\cdot d := p^*C\cdot q^*D$ is well-defined. This defines an intersection form on $Z_C(\mathbb{P}^2)$. If $e_p\in Z_C(\mathbb{P}^2)$ denotes the class of the exceptional divisor of a point $p\in\mathcal{B}(\mathbb{P}^2)$, then $e_p^2 = -1$ and if $p, q\in\mathbb{B}(\mathbb{P}^2)$ are distinct points, then $e_p\cdot e_q = 0$. So, the intersection form has signature $(1,\infty)$.

We now tensor $Z_C(\mathbb{P}^2)$ with $\mathbb{R}$ to get a vector space:

$$Z_C(\mathbb{P}^2)_{\mathbb{R}} := (NS(\mathbb{P}^2)\otimes\mathbb{R}) \oplus \left(\oplus_{p\in\mathcal{B}(\mathbb{P}^2)}\mathbb{R}e_p\right),$$

The intersection form on $Z_C(\mathbb{P}^2)$ extends naturally to an intersection form on $Z_C(\mathbb{P}^2)$. Consider the ℓ^2-completion of $Z_C(\mathbb{P}^2)_{\mathbb{R}}$

$$\mathcal{Z}(\mathbb{P}^2) := \left\{ \bar{D} + \sum_{p\in\mathcal{B}(\mathbb{P}^2)} \lambda_p e_p \mid \lambda_p \in \mathbb{R},\ \sum_{p\in\mathcal{B}(\mathbb{P}^2)} \lambda_p^2 < \infty,\ \overline{D} \in NS(\mathbb{P}^2)\otimes\mathbb{R} \right\},$$

It is called *Picard-Manin space* of $\mathbb{P}^2$ and we can again think

$$\mathcal{Z}(\mathbb{P}^2) \subseteq (NS(\mathbb{P}^2)\otimes\mathbb{R}) \oplus \left(\sum_{p\in\mathcal{B}(\mathbb{P}^2)} \mathbb{R}e_p \right).$$

The intersection form on $Z_C(\mathbb{P}^2)_{\mathbb{R}}$ extends naturally to an intersection form on $\mathcal{Z}(\mathbb{P}^2)$ of signature $(1,\infty)$ and makes $\mathcal{Z}(\mathbb{P}^2)$ a Hilbert space. It cuts out a hyperbolic space as follows: if $\ell \in Z_C(\mathbb{P}^2)$ is the class of a line in $\mathbb{P}^2$, we set

$$\mathbb{H}^\infty(\mathbb{P}^2) := \{c \in \mathcal{Z}(\mathbb{P}^2) \mid c\cdot c = 1,\ c.\ell > 0\}$$

and endow it with the metric $d(c_1, c_2) = \operatorname{arcosh}(c_1 \cdot c_2)$, $c_1, c_2 \in \mathbb{H}^\infty(\mathbb{P}^2)$, which makes $\mathbb{H}^\infty(\mathbb{P}^2)$ a hyperbolic space of infinite dimension; the condition $c \cdot c = 1$ cuts out two hyperbolic spaces and the condition $c \cdot \ell > 0$ chooses one connected component.

Now, let's define the action of $\operatorname{Bir}(\mathbb{P}^2)$ on our hyperbolic space. The action should reflect that if f is a local isomorphism at a point p, then f sends the class of the exceptional divisor of p to the class of the exceptional divisor of $f(p)$; we want $f_\#(e_p) = e_{f(p)}$. If L is a general line in $\mathbb{P}^2$, then f sends L onto a curve C passing through all base-points $p_1, \dots, p_n$ of f^{-1}, and the degree $\deg(f) := \deg(C)$ and its multiplicities $m_{p_i}(f)$ at the base-points of f^{-1} do not depend on the choice of L. Then f should send the class $\bar{L}$ onto the class $\bar{C}$. In other words, we want $f_\#(\bar{L}) = \deg(f)\bar{L} - m_{p_1}(f)e_{p_1} - \cdots - m_{p_n}(f)e_{p_n}$.

We will not go into more detail of the definition of the action, which can be found for instance in [34, §1.2.3]. What is important is that because of the way the intersection lifts onto blow-ups of points, we have $f_\#(c_1) \cdot f_\#(c_2) = c_1 \cdot c_2$ for any $c_1, c_2 \in \mathcal{Z}(\mathbb{P}^2)$. So, the $\mathbb{H}^\infty(\mathbb{P}^2)$ and its hyperbolic metric are invariant under the action of $\operatorname{Bir}(\mathbb{P}^2)$ on $\mathcal{Z}(\mathbb{P}^2)$. We have reached our goal: $\operatorname{Bir}(\mathbb{P}^2)$ acts faithfully on the infinite hyperbolic space $\mathbb{H}^\infty(\mathbb{P}^2)$ by isometries.

One can check that $\mathbb{H}^\infty(\mathbb{P}^2)$ is a complete CAT(0) metric space. This gives rise to the notion of boundary at infinity, which we denote by $\partial\mathbb{H}^\infty(\mathbb{P}^2)$. Details on this may be found for instance in [6, Chapter II.8].

This is a good moment to notice that while the construction of $\mathcal{Z}(\mathbb{P}^2)$ generalizes to higher dimension, the construction of the hyperbolic space $\mathbb{H}^\infty(\mathbb{P}^2)$ does not, because there is no bilinear intersection form of divisors in higher dimension. The articles [15, 16] of N.-B. Dang and C. Favre indicate how in higher dimension the hyperbolic space may be replaced by a Banach space.

The elements of $\operatorname{Bir}(\mathbb{P}^2)$ satisfy analogues of classical properties of isometries of finite dimensional hyerbolic spaces. For $f \in \operatorname{Bir}(\mathbb{P}^2)$, let $L(f) := \inf\{d(f(x), x) \mid x \in \mathbb{H}^\infty(\mathbb{P}^2)\}$. Then f is exactly one of the following three types:

- *elliptic*, if $L(f) = 0$ and the infinum is achieved, and then f has a fixed point in $\mathbb{H}^\infty(\mathbb{P}^2)$;
- *parabolic*, if $L(f) = 0$ and the infinum is not achieved, and then f fixes exactly one point on $\partial\mathbb{H}^\infty(\mathbb{P}^2)$;
- *hyperbolic/loxodromic*, if $L(f) > 0$. In this case, $\operatorname{Ax}(f) := \{x \in \mathbb{H}^\infty(\mathbb{P}^2) \mid L(f) = d(f(x), x)\}$ is a geodesic and f acts by translation on it with translation length $L(f)$. Moreover, f has exactly two fixed points on $\partial\mathbb{H}^\infty(\mathbb{P}^2)$, one is attractive, the other repulsive.

The property of being elliptic/parabloic/loxodromic is related to the *dynamical degree* $\lambda(f) = \lim_{n\longrightarrow\infty} \deg(f^n)^{1/n}$ of f. The relation between the dynamical degree and the isometry type is the following [8, Theorem 3.7]:

- elliptic $\Longleftrightarrow$ growth of $\deg(f^n)$ is bounded
- parabolic $\Longleftrightarrow$ growth of $\deg(f^n)$ is linear or quadratic
- hyperbolic/loxodromic $\Longleftrightarrow$ growth of $\deg(f^n)$ is exponential.

In the story of normal subgroups of $\mathrm{Bir}(\mathbb{P}^2)$, hyperbolic elements of $\mathrm{Bir}(\mathbb{P}^2)$ play a main role.

2.1.2. *Small simplification and normal subgroups of* $\mathrm{Bir}(\mathbb{P}^2)$. We have established that $\mathrm{Bir}(\mathbb{P}^2)$ acts by isometries on a hyperbolic space $\mathbb{H}^\infty(\mathbb{P}^2)$. This is a set-up for using geometric group geometry, more specifically small simplification, to show the existence of normal subgroups of $\mathrm{Bir}(\mathbb{P}^2)$.

Let $g \in \mathrm{Bir}(\mathbb{P}^2)$. Every element of the normal subgroup $\langle\langle g\rangle\rangle \subset \mathrm{Bir}(\mathbb{P}^2)$ generated by g is of the form $h_r \cdots h_1$, where each h_i is a conjguate of g or g^{-1}, that is, $h_i = s_i g s_i^{-1}$ or $h_i = s_i g^{-1} s_i^{-1}$ for some $s_i \in \mathrm{Bir}(\mathbb{P}^2)$. If g is hyperbolic, this means that each h_i has fixed axis $s_i\mathrm{Ax}(g)$.

In [11], S. Cantat and S. Lamy define the following property if $\mathbf{k}$ is algebraically closed. A hyperbolic element $g \in \mathrm{Bir}(\mathbb{P}^2)$ is *tight* if the following two conditions hold:

- there is some $B > 0$ such that if $f \in \mathrm{Bir}(\mathbb{P}^2)$ and $f(\mathrm{Ax}(g))$ contains two points at distance B which are at distance at most one from $\mathrm{Ax}(g)$, then $f(\mathrm{Ax}(g)) = \mathrm{Ax}(g)$.
- If $f \in \mathrm{Bir}(\mathbb{P}^2)$ and $f(\mathrm{Ax}(g)) = \mathrm{Ax}(g)$, then $fgf^{-1} = g^{\pm 1}$.

In less precise words, a hyperbolic element of $g \in \mathrm{Bir}(\mathbb{P}^2)$ is tight if any element of $\mathrm{Bir}(\mathbb{P}^2)$ sending $\mathrm{Ax}(g)$ to an axis close to $\mathrm{Ax}(g)$ over a long period of time is conjugate to g or g^{-1}. This notion allowed them to show the following theorem, which is a variant of small simplification:

Theorem 2.1 ([11]). *Let* $\mathbf{k}$ *be an algebraically closed field. If* $g \in \mathrm{Bir}(\mathbb{P}^2)$ *is tight, then there exists an integer* $n \geqslant 1$ *such that any nontrivial element* $h \in \langle\langle g^n\rangle\rangle$ *satisfies* $\deg(h) \geqslant \deg(g^n)$. *In particular,* $\langle\langle g^n\rangle\rangle$ *is a strict subgroup of* $\mathrm{Bir}(\mathbb{P}^2)$.

Theorem 2.2 ([11]). *If* $\mathbf{k}$ *is algebraically closed, then* $\mathrm{Bir}(\mathbb{P}^2)$ *contains tight elements. In particular,* $\mathrm{Bir}(\mathbb{P}^2)$ *is not simple.*

In fact, they showed that any general element in $\mathrm{Bir}(\mathbb{P}^2)$ is tight, if $\mathbf{k}$ is algebraically closed [11, Theorem 5.2]. A generalisation of the above theorem for a larger family of base-fields was obtained by N. I. Shepherd-Barron:

Theorem 2.3 ([37]). *Let* $\mathbf{k}$ *be a field and* $g \in \mathrm{Bir}(\mathbb{P}^2)$ *a hyperbolic element. Suppose that* $L(g)$ *is not the logarithm of a quadratic unit; if* $\mathrm{char}(\mathbf{k}) = p > 0$, *assume also that* $\mathbf{k}$ *is algebraic and* $L(g)$ *is not an integral multiple of* $\log(p)$. *Then some power of* g *is tight. In particular,* $g \notin \langle\langle g^N\rangle\rangle$ *for sufficiently large* $N \in \mathbb{N}$, *and so* $\langle\langle g^N\rangle\rangle$ *is a proper normal subgroup of* $\mathrm{Bir}(\mathbb{P}^2)$.

Theorem 2.4 ([37]). *Let* $\mathbf{k}$ *be a finite field and* $g \in \mathrm{Bir}(\mathbb{P}^2)$ *a hyperbolic element. Then* g *is tight and* $g \notin \langle\langle g^N\rangle\rangle$ *for sufficiently large* $N \in \mathbb{N}$.

When A. Lonjou set out to show non-simplicity of the plane Cremona group over any base-field $\mathbf{k}$, she used a notion common in geometric group theory. We say that a hyperbolic isometry g of $\mathbb{H}^\infty(\mathbb{P}^2_{\mathbf{k}})$ *acts discretely on its axis* $\mathrm{Ax}(g)$ or to have *property WPD* (weak proper discontinuity), if there exists $x \in \mathbb{H}^\infty(\mathbb{P}^2_{\mathbf{k}})$ such that for any $\varepsilon > 0$, there exists an integer $n > 0$ such that

$$\mathrm{Fix}_\varepsilon\{x, g^n(x)\} := \{f \in \mathrm{Bir}(\mathbb{P}^2) \mid d(x, f(x)) < \varepsilon,\ d(g^n(x), f(g^n(x))) < \varepsilon\}$$

is finite [33, §1.2]. In very unprecise words, a hyperbolic element $g \in \mathrm{Bir}(\mathbb{P}^2)$ has property WPD if there are only a finite number of elements of $\mathrm{Bir}(\mathbb{P}^2)$ that move $\mathrm{Ax}(g)$ onto an axis close to $\mathrm{Ax}(g)$. The definition is equivalent to the one where we replace "there exists $x \in \mathbb{H}^\infty(\mathbb{P}^2_{\mathbf{k}})$ such that for all $\varepsilon > 0$..." by "for all $x \in \mathbb{H}^\infty(\mathbb{P}^2_{\mathbf{k}})$ and for any $\varepsilon > 0$..." [33, Lemma 5].

The notion of *tight* and *satisfying property WPD* are related, but see [28, Examples 5.10 and 5.11] for examples of elements of $\mathrm{Bir}(\mathbb{P}^2)$ that are tight but do not satisfy property WPD and examples of elements that are not tight but satisfy property WPD.

The notion of having property WPD was used by F. Dahmani, V. Guirardel and D. Osin in [14] to show the existence of normal subgroups:

Theorem 2.5 ([14])**.** *Let $C \in \mathbb{R}_{>0}$ and let G be a group acting by isometries on a hyperbolic space X in the sense of Gromov (see [14, §3] and [21]) and let $g \in G$ be a hyperbolic element. If G acts discretly on the axis $\mathrm{Ax}(g)$ of g, then there is an integer $n \geqslant 1$ such that for any nontrivial element $h \in \langle\langle g^n \rangle\rangle$ we have $L(h) > C$, where $L(h)$ is the length of the translation by h on $\mathrm{Ax}(g)$.*

In particular, for n large enough, $\langle\langle g^n \rangle\rangle$ is a strict subgroup of G. Moreover, $\langle\langle g^n \rangle\rangle$ is a free group.

A. Lonjou showed that $\mathrm{Bir}(\mathbb{P}^2)$ contains hyperbolic elements with property WPD:

Theorem 2.6 ([33])**.** *Let $m \geqslant 2$ and $\mathbf{k}$ be a field of characteristic $p \geqslant 0$ that does not divide m. Then $\mathrm{Bir}(\mathbb{P}^2)$ acts discretely on the axis of $g_m\colon (x,y) \mapsto (y, y^m - x)$. In particular, there exists an integer $n \geqslant 1$ such that $\langle\langle g^n \rangle\rangle$ is a strict subgroup of $\mathrm{Bir}(\mathbb{P}^2)$.*

For more on small simplification and Cremona groups, see [11, 28]. The normal subgroups $\langle\langle g^n \rangle\rangle$ of $\mathrm{Bir}(\mathbb{P}^2)$ found by A. Lonjou are large, since they are free groups. However, also their quotients are large: by [14], they are SQ-universal, meaning that any finitely generated group embeds into a quotient of $\mathrm{Bir}(\mathbb{P}^2)/\langle\langle g^n \rangle\rangle$.

2.1.3. *Classification of normal subgroups of* $\mathrm{Bir}(\mathbb{P}^2)$ *generated by one element.* The normal subgroups generated by one element are almost classified by S. Cantat, V. Guirardel and A. Lonjou and also separately by N. I. Shepherd-Barron.

A *monomial* element of $\mathrm{Bir}(\mathbb{P}^2)$ is of the form $(x,y) \mapsto (x^a y^b, x^c y^d)$, $a,b,c,d \in \mathbb{Z}$, $ad - bc \neq 0$, and they make up a group isomorphic to $\mathrm{GL}_2(\mathbb{Z})$. It is the group of elements in $\mathrm{Bir}(\mathbb{P}^2)$ commuting with the canonical torus action on $\mathbb{P}^2$.

If $\mathrm{char}(\mathbf{k}) = p > 0$, there are special transformations of $\mathbb{P}^2$: we define the group $p\,\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ of *p-automorphisms* to be the normaliser in $\mathrm{Aut}(\mathbb{A}^2_{\mathbf{k}})$ of the group of translations $t_{u,v}\colon (x,y) \mapsto (x+u, y+v)$. It has the following description: let $\mathrm{Fr}\colon t \mapsto t^p$ be the Frobenius endomorphism of $\mathbf{k}$. Let $\mathbf{k}[\mathrm{Fr}]$ be the (non-commutative) algebra of polynomials of the form $\sum a_i t^{p^j} = \sum a_i \mathrm{Fr}^j(t)$, where Fr^j is the j-th composition of Fr. The $\mathbf{k}$-points of the group of algebraic automorphisms of $\mathbb{G}_{a,\mathbf{k}} \times \mathbb{G}_{a,\mathbf{k}}$ coincides with $\mathrm{GL}_2(\mathbf{k}[\mathrm{Fr}])$ and $(p\,\mathrm{Aut}(\mathbb{A}^2))(\mathbf{k}) \simeq (\mathbb{G}_{a,\mathbf{k}}(\mathbf{k}) \times \mathbb{G}_{a,\mathbf{k}}(\mathbf{k})) \rtimes \mathrm{GL}_2(\mathbf{k}[\mathrm{Fr}])$ [10, §3.3.1].

Theorem 2.7 ([10])**.** *Let $\mathbf{k}$ be an algebraically closed field and let $f \in \mathrm{Bir}(\mathbb{P}^2)$ be of infinite order. The following are equivalent:*

- *there exists $n \geqslant 1$ such that $\langle\langle f^n \rangle\rangle$ is a strict normal subgroup of $\mathrm{Bir}(\mathbb{P}^2)$.*
- *The growth of $\deg(f^n)$ is quadratic or f is a hyperbolic element not conjugate to a monomial transformation with or to a p-automorphism if $\mathrm{char}(\mathbf{k}) = p > 0$.*

2.2. **Non-trivial quotients from the plane Cremona group to $\mathbb{Z}/2$ and the Sarkisov program.** In this section, we look at the second type of construction of normal subgroups of Cremona groups of the plane. We also glance at the tool used for it, the so-called Sarkisov program. In this section, the base field matters and so we will again write $\mathbb{P}^2_{\mathbf{k}}$ and $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$.

2.2.1. *Non-trivial quotients* $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow \mathbb{Z}/2$. When looking for normal subgroups, another approach is to try to cook up non-trivial quotient maps whose kernel is non-trivial. For the Cremona group over an algebraically closed field, it is not straight forward to find one, as the following statement indicates.

Lemma 2.8. *Let* $\mathbf{k}$ *be an algebraically closed field. Then the following hold:*

(1) Any homomorphism $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow G$ *to a finite group* G *is trivial.*
(2) The commutator subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ *is equal to* $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$*.*

Proof. If a homomorphism $\varphi\colon \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow G$ to a finite group G exists, then $\ker(\varphi)$ contains a non-trivial element $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}}) \simeq \mathrm{PGL}_3(\mathbf{k}) = \mathrm{PSL}_3(\mathbf{k})$. Since the latter is a simple group, $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ is contained in $\ker(\varphi)$. Moreover, the involutions $h\colon (x,y) \mapsto (1-x, 1-y)$ and $\sigma\colon (x,y) \dashrightarrow (\frac{1}{x}, \frac{1}{y})$ satisfy the relation $(\sigma h)^3 = \mathrm{id}$. It follows that $\sigma = (h\sigma)h(h\sigma)^{-1}$ is contained in $\ker(\varphi)$ as well. By the Noether-Castelnuov theorem [12], the set $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}}) \cup \{\sigma\}$ generates $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ when $\mathbf{k}$ is algebraically closed. So, φ is trivial. Moreover, the commutator subgroup D of $\mathrm{Bir}(\mathbb{P}^2)$ contains a non-trivial automorphism, so Lemma 1.1 implies that $D = \mathrm{Bir}(\mathbb{P}^2)$. □

However, over non-closed fields, we can find non-trivial homomorphisms $\mathrm{Bir}(\mathbb{P}^2) \longrightarrow \mathbb{Z}/2$. For instance, the group $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{R}})$ is generated by $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{R}}) \simeq \mathrm{PGL}_3(\mathbb{R})$, the standard Cremona involution $\sigma\colon (x,y) \dashrightarrow (\frac{1}{x}, \frac{1}{y})$, the involution at the circle $\tau\colon (x,y) \dashrightarrow (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ and a family of transformations of $\mathbb{P}^2$ of degree five, called standard quintic transformations with six non-real base-points of multiplicity two [2].

Theorem 2.9 ([40])**.** *The commutator subgroup* D *of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{R}})$ *is the smallest normal subgroup of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{R}})$ *containing* $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{R}})$ *and* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{R}})/D \simeq \bigoplus_I \mathbb{Z}/2$*, where* I *is an uncountable set.*

The abelianisation morphism $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{R}}) \longrightarrow \bigoplus_I \mathbb{Z}/2$ sends $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{R}})$ and the quadratic maps σ and τ to zero, while the generators of $\bigoplus_I \mathbb{Z}/2$ are images of standard quintic transformations. There is a geometric intuition for this: there are essentially two types of pencils of conics in $\mathbb{P}^2_{\mathbb{R}}$ - one is the pencil of lines through a real point of $\mathbb{P}^2_{\mathbb{R}}$, the other is the pencil of conics through two pairs of non-real conjugate points in $\mathbb{P}^2_{\mathbb{R}}$. The maps σ and τ preserve a pencil of lines through $(0:0:1)$. The map τ and the standard quintic transformations preserve the pencil of conics through $p_1 := (1:i:0), \overline{p_1} = (1:-i:0), p_2 := (0:1:i), \overline{p_2} = (0:1:-i)$. Let $Y_5 \longrightarrow \mathbb{P}^2_{\mathbb{R}}$ be the blow up of $p_1, \overline{p_1}, p_2, \overline{p_2}$ in $\mathbb{P}^2_{\mathbb{R}}$. It carries the conic bundle structure $Y_5 \longrightarrow \mathbb{P}^1_{\mathbb{R}}$ induced by the pencil of conics through $p_1, \overline{p_1}, p_2, \overline{p_2}$ and it has three singular fibres. The contraction of the strict transform of the line through $p_1, \overline{p_1}$ yields a morphism $Y_5 \longrightarrow X_6$ over $\mathbb{P}^1_{\mathbb{R}}$ to a del Pezzo surface X_6 of degree 6. The conic fibration $X_6 \longrightarrow \mathbb{P}^1_{\mathbb{R}}$ is a minimal conic fibration with two singular fibres. We can view the standard quintic transformations as elementary transformations of the conic fibration $X_6 \longrightarrow \mathbb{P}^1_{\mathbb{R}}$.

This construction generalises to non-closed perfect fields. Let $\mathbf{k}$ be a perfect field and $\overline{\mathbf{k}}$ its algebraic closure. If X is a surface, then the Galois group $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ acts on $X_{\overline{\mathbf{k}}} := X \times_{\mathrm{Spec}(\mathbf{k})} \mathrm{Spec}(\overline{\mathbf{k}})$ by acting on the second factor. There is a bijection between varieties over $\mathbf{k}$ and $\mathbf{k}$-structures on $X_{\overline{\mathbf{k}}}$, and we refer to [5] for precise details. Let $p \in X$ be a closed point. Then $p_{\overline{\mathbf{k}}} = p \times_{\mathrm{Spec}(\mathbf{k})} \mathrm{Spec}(\overline{\mathbf{k}})$ is a finite union of points in $X_{\overline{\mathbf{k}}}$ and they are permuted by $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$. The degree of p is the number of connected components of $p_{\overline{\mathbf{k}}}$. Points $p_1, \ldots, p_n$ are in general position if the connected components of $(p_1)_{\overline{\mathbf{k}}} \cup \cdots \cup (p_n)_{\overline{\mathbf{k}}}$ are in general position over $\overline{\mathbf{k}}$.

If $[\overline{\mathbf{k}}:\mathbf{k}]>2$, there may moreover exist pencils of conics through a point p of degree 4 in general position. Let $X_5\longrightarrow\mathbb{P}^2_{\mathbf{k}}$ be the blow up of p. Then the pencil of conics through p induces a conic bundle structure $X_5\longrightarrow\mathbb{P}^1_{\mathbf{k}}$ which is minimal.

J. Schneider shows that there are no relations between elementary transformations of $X_5/\mathbb{P}^1_{\mathbf{k}}$, of $X_6/\mathbb{P}^1_{\mathbf{k}}$ and of Hirzebruch surfaces with a base-point of degree$\geqslant 16$. They also show that there are no relations between elementary transformations of distinct isomorphism classes of fibrations of type $X_5/\mathbb{P}^1_{\mathbf{k}}$ and between elementary transformations of distinct isomorphism classes of conic fibrations of type $X_6/\mathbb{P}^1_{\mathbf{k}}$. She uses this to construct the following quotient of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$.

Let J_5 be the set of pencils of conics passing through a point of degree 4 in general position up to automorphism of $\mathbb{P}^2_{\mathbf{k}}$. Let J_6 the set of pencils of conics passing through points of degree 2 in $\mathbb{P}^2_{\mathbf{k}}$ in general position up to automorphisms of $\mathbb{P}^2_{\mathbf{k}}$.

Theorem 2.10 ([36]). *Let $\mathbf{k}$ be a perfect field such that $[\overline{\mathbf{k}}:\mathbf{k}]>2$. Then there exists a surjective homomorphism*

$$\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})\longrightarrow\left(\bigoplus_{I_0}\mathbb{Z}/2\right)*\left(\mathop{\ast}_{J_5}\bigoplus_{I}\mathbb{Z}/2\right)*\left(\mathop{\ast}_{J_6}\bigoplus_{I}\mathbb{Z}/2\right)$$

where I_0 is the infinite set of points in $\mathbb{P}^1_{\mathbf{k}}$ of degree$\geqslant 16$ and I is the set of integers $n\geqslant 8$ such that there exists an irreducible polynomial in $\mathbf{k}[x]$ of degree $2n+1$.

The first free factor is generated by images of elementary transformations of Hirzebruch surfaces, more precisely by the images of $f_p\colon(x,y)\dashrightarrow(xp(y),y)$ where $p\in\mathbf{k}[y]$ is irreducible of degree$\geqslant 16$ and f_p is mapped to the element whose non-zero entries are indexed by the roots of p. The second free factor is generated by images of elementary transformations of conic bundles of type $X_5/\mathbb{P}^1_{\mathbf{k}}$. The third free factor is generated by images of elementary transformations of conic bundles of type $X_6/\mathbb{P}^1_{\mathbf{k}}$.

The projection onto the first free factor can also be seen as follows: consider the subgroup

$$\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})\supset\left\{(x,y)\dashrightarrow\left(\frac{a(y)x+b(y)}{c(y)x+d(y)},y\right)\mid a,b,c,d\in\mathbf{k}(y),ad-bc\neq 0\right\}\simeq\mathrm{PGL}_2(\mathbf{k}(y))$$

and the homomorphism of groups

$$\varphi\colon\ \mathrm{PGL}_2(\mathbf{k}(y))\xrightarrow{det}\mathbf{k}(y)^*/(\mathbf{k}(y)^*)^2\simeq\bigoplus_{P}\mathbb{Z}/2\xrightarrow{pr_{I_0}}\bigoplus_{I_0}\mathbb{Z}/2$$

where P is the set of closed points in $\mathbb{P}^1_{\mathbf{k}}$ and p_{I_0} the projection onto I_0. The isomorphism $\mathbf{k}(y)^*/(\mathbf{k}(y)^*)^2\simeq\bigoplus_P\mathbb{Z}/2$ is given as follows: if $\mathbf{k}(y)^*\ni f=\prod f_i^{e_i}$ is the decompsition into irreducible polynomials f_i with $e_i\in\mathbb{Z}$, then the class $[f]$ of f maps to $\sum_{\mathrm{div}(f_i)\in P}[e_i]$. It turns out that φ is the restriction of the homomorphism in Theorem 2.10 to the subgroup $\mathrm{PGL}_2(\mathbf{k}(y))$.

There is a similar interpretation for the projection onto the sum $\bigoplus_I\mathbb{Z}/2$ in the other free factors by using the group of birational maps of a conic bundle $X_6/\mathbb{P}^1_{\mathbf{k}}$ (resp. $X_5/\mathbb{P}^1_{\mathbf{k}}$) preserving the fibration and inducing the identity on the base.

There have not been any results on properties of the kernel of the homomorphisms Theorem 2.10 up to this moment.

A similar quotient was obtained in [30], where the construction does not focus on conic fibrations. Consider a point of p degree 8 in $\mathbb{P}^2_{\mathbf{k}}$ in general position (it always exists [30]). There is a pencil of cubic curves through p and it has a ninth base-point q, which must be rational. Let $\pi\colon X_1\longrightarrow\mathbb{P}^2_{\mathbf{k}}$ be the blow-up of p. Each smooth cubic in the pencil is an elliptic curve and we choose q to be zero on each elliptic curve. The map $x\mapsto -x$ on each elliptic curve induces an isomorphism ι of oder two of the surface X_1. Another way to see this, is as follows: the del Pezzo surface X_1 embeds as hypersurface of degree

6 in $\mathbb{P}(1,1,2,3)$. The projection $X_1 \longrightarrow \mathbb{P}(1,1,2)$ is a double cover of a quadric cone and ι is its Deck-transofrmation. The birational map $\beta := \pi \circ \iota \circ \pi^{-1}$ of $\mathbb{P}^2_{\mathbf{k}}$ is called Bertini involution.

Theorem 2.11 ([30])**.** *Let* $\mathbf{k}$ *be a perfect field with a Galois extension of degree* 8*. Then there exists a surjective morphism* $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow \ast_B \mathbb{Z}/2$*, where the cardinality of* B *is at least the cardinality of* $\mathbf{k}$*. Moreover, this homomorphism has a section.*

The free product is generated by images of Bertini involutions and B is the set of Bertini involutions of $\mathbb{P}^2_{\mathbf{k}}$ up to conjugation with automorphisms. We do know the kernel of the above group homomorphism and we give its description at the end of the next section after a glance at the Sarkisov program.

For some time it was unclear why the quotients we are able to construct are always free products of sums of $\mathbb{Z}/2$. This was answered when S. Lamy and J. Schneider showed that for any perfect field $\mathbf{k}$, the group $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is generated by involutions [29]. This is false in higher dimension [31, 3].

2.2.2. *An interlude on the Sarkisov program.* Of course, to prove the statement of Theorem 2.10 it is not enough to play with relations among elementary transformations of conic fibrations, because $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is not generated by elementary transformations. The proof uses the so-called Sarkisov program as developped by A. Corti and V. A. Iskovskikh after an idea of Sarkisov [13, 24]. *Sarkisov links* are a generalisation of elementary transformations of conic fibrations and are birational maps χ as in Figure 1, where $*$ is a point,

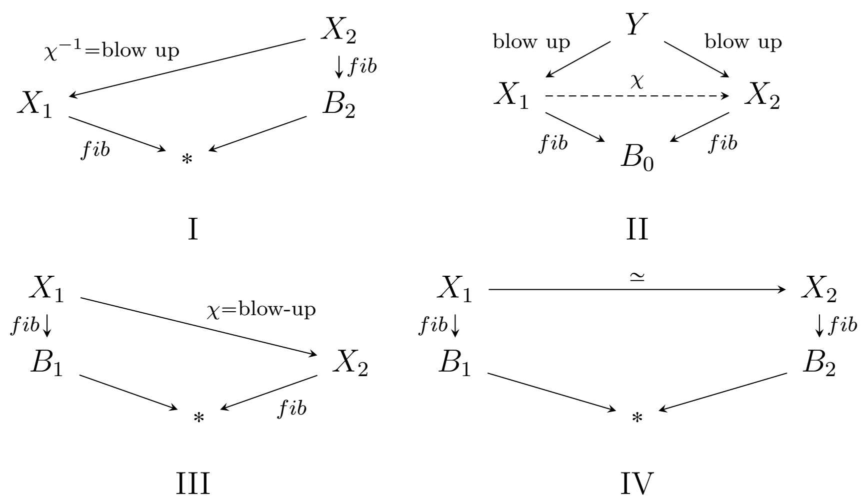


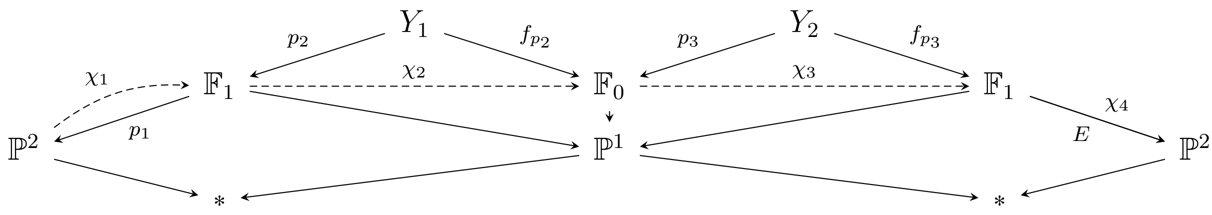
FIGURE 1. The four types Sarkisov links in dimension 2.

fib stands a fibration $X \longrightarrow B_i$ with connected fibres such that $-K_X$ is relatively ample and the relative Picard rank is $\rho(X/B_i) = 1$, and all non-horizontal morphisms have relative Picard rank 1. If B_i is a curve, this means the morphism is a minimal conic fibration. If $B_i = *$ is a point, this means that X is a minimal del Pezzo surface. In short, fib stands for Mori fibre space in dimension 2.

The blow-up of a rational point in $\mathbb{P}^2_{\mathbf{k}}$ is an example of a Sarkisov link of type I, and its inverse is an example of a Sarkisov link of type III. The exchange of the two fibrations of $\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}}$ is an example of a Sarkisov link of type IV. Type II are elementary transformations between minimal conic bundles if B_0 is a curve. A Bertini involution with a base-point of degree 8 is an example of a link of type II where B_0 is a point.

Theorem 2.12 ([13, 24]). *Let* $\mathbf{k}$ *be a perfect field. Any birational map of* $\mathbb{P}^2_{\mathbf{k}}$ *is a composition of Sarkisov links.*

Let us illustrate a well-known composition of Sarkisov links over an arbitrary perfect field $\mathbf{k}$. Consider the involution $\sigma\colon [x:y:z] \dashmapsto [yz:xz:xy]$ of $\mathbb{P}^2_{\mathbf{k}}$. It has three rational base-points, namely $p_1 := [1:0:0]$, $p_2 := [0:1:0]$, $p_3 := [0:0:1]$, and contracts the three lines passing through any two of the three points. We can write $\sigma = \chi_4 \circ \cdots \circ \chi_1$, with $\chi_1, \ldots, \chi_4$ the Sarkisov links in the commutative diagram below, where the blow-ups are marked with the point that is blown up or strict transform f_p of the fibre containing the point p that is contracted, or the exceptional curve $E \subset \mathbb{F}_1$ that is contracted.



Let $\mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}})$ be the groupoid generated by Sarkisov links between rational surfaces. By Theorem 2.12, it contains the Cremona group $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$. It has a nice presentation in terms of generators and generating relations [30] and the generating relations are called *elementary relations*. We will not go further into this, but in [29], S. Lamy and J. Schneider list all elementary relations of Sarkisov links between rational surfaces over a perfect field.

A core idea for the homomorphism in Theorem 2.10 is the following: let $\mathrm{CB}(\mathbb{P}^2_{\mathbf{k}})$ be the set of classes of rational minimal conic fibrations up to birational maps preserving the conic bundle structure. Let $X \longrightarrow \mathbb{P}^1_{\mathbf{k}}$ be a rational minimal conic fibration in the class $C \in \mathrm{CB}(\mathbb{P}^2_{\mathbf{k}})$. Two elementary transformations $\chi, \chi'\colon X \dashrightarrow X'$ over $\mathbb{P}^1_{\mathbf{k}}$ (i.e. Sarkisov links of type II over $B = \mathbb{P}^1_{\mathbf{k}}$) are equivalent if their base-points have the same degree. We denote by $M(C)$ the equivalence class of elementary transformations of the class C. Using the list of elementary relations between Sarkisov links, J. Schneider proves the following:

Theorem 2.13 ([36]). *Let* $\mathbf{k}$ *be a perfect field. There is a homomorphism of groupoids*

$$\mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow \mathop{\ast}_{C \in \mathrm{CB}(\mathbb{P}^2_{\mathbf{k}})} \Big(\bigoplus_{\chi \in M(C)} \mathbb{Z}/2 \Big)$$

that sends each Sarkisov link of type II between conic fibrations with base-point of degree $\geqslant 16$ *onto the generator indexed by its equivalence class, and all other Sarkisov links and all automorphisms onto zero.*

Then they compose the homomorphism of groupoids with the projection onto the free factors from Theorem 2.10 and show that this composition is surjective.

The same approach is taken in [30], where it is shown that $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \simeq G \ast (\ast_B \mathbb{Z}/2)$, where $G \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is the subgroup generated by $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ and compositions $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of Sarkisov links that are not Bertini involutions with a base-point of degree 8. The homomorphism in Theorem 2.11 is the projection forgetting G.

3. Normal subgroups in dimension $\geqslant 3$

In this section, we look at construction of normal subgroups of $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ for $n \geqslant 3$. The strategy is to construct a non-trivial morphism of groups $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}) \longrightarrow G$ for some group G whose kernel is non-trivial. The sections are ordered by $G = \mathbb{Z}/2$, $G = \mathbb{Z}$ and $G = \mathbb{Z}/3$ and $G = \mathcal{F}(\mathbb{C})$.

3.1. Quotients of Cremona groups via conic bundles.

The concept of Sarkisov links generalises to higher dimension. We state the following theorem without going into the generalisation Sarkisov links to higher dimension. Definitions can be found in [1, 22]. Theorem 2.12 generalises to higher dimension as well:

Theorem 3.1 ([22])**.** *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic zero and let* $n \geqslant 3$. *Then any birational map of* $\mathbb{P}^n_{\mathbf{k}}$ *is a composition of Sarkisov links.*

Let $\mathrm{BirMori}(\mathbb{P}^n_{\mathbf{k}})$ be the groupoid generated by Sarkisov links between rational n-folds. It contains $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ by the above theorem. [1] gives a presentation in terms of generators and generating relations for $\mathrm{BirMori}(\mathbb{P}^n_{\mathbf{k}})$ called; generating relations are called *elementary relations* and they list the ones including a Sarksiov link between conic fibrations. We have the following analogon of Theorem 2.13. The gonality of a curve C is the minimal degree of a dominant morphism $C \longrightarrow \mathbb{P}^1_{\mathbf{k}}$. The covering gonality d of a variety Γ is the minimal positive constant such that through any general point of Γ passes a curve in Γ of gonality d.

Theorem 3.2 ([1])**.** *Let* $\mathbf{k} \subset \mathbb{C}$ *be a subfield. For each* $n \geqslant 3$ *there exists a constant* $d_n > 0$ *and a homomorphism of groupoids*

$$\mathrm{BirMori}(\mathbb{P}^n_{\mathbf{k}}) \longrightarrow \mathop{\ast}_{C \in \mathrm{CB}(\mathbb{P}^n_{\mathbf{k}})} \Big(\bigoplus_{\chi \in M(C)} \mathbb{Z}/2 \Big)$$

that sends each Sarkisov link of type II between conic fibrations with base-locus of covering gonality $\geqslant d_n$ *over* $\mathbb{C}$ *onto the generator indexed by its equivalence class, and all other Sarkisov links and all automorphisms onto zero.*

Projecting on the equivalence class of the conic bundle $\mathbb{P}^{n-1}_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}} \longrightarrow \mathbb{P}^{n-1}_{\mathbf{k}}$ yields an analogon of Theorem 2.10 in higher dimension, which shows that all Cremona groups in higher dimension are not simple.

Theorem 3.3 ([1])**.** *Let* $n \geqslant 3$ *and let* $\mathbf{k} \subset \mathbb{C}$ *be a subfield. There is a surjective homomorphism of groups*

$$\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}) \longrightarrow \bigoplus_I \mathbb{Z}/2,$$

where the indexing set I *has the same cardinality as* $\mathbf{k}$ *and it restriction to the subgroup of birational dilatations* $(x_1, \dots, x_n) \dashrightarrow (x_1\alpha(x_2, \dots, x_n), x_2, \dots, x_n)$ *is surjective. In particular,* $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ *is not simple for* $n \geqslant 3$.

The geometric meaning of the homomorphism is again as follows: consider the subgroup

$$\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}) \supset \left\{ (x, \underline{y}) \dashrightarrow \left(\frac{a(\underline{y})x + b(\underline{y})}{c(\underline{y})x + d(\underline{y})}, \underline{y} \right) \mid a, b, c, d \in \mathbf{k}(\underline{y}), ad - bc \neq 0 \right\} \simeq \mathrm{PGL}_2(\mathbf{k}(\underline{y})),$$

where $\underline{y} = (y_2, \dots, y_n)$, and the homomorphism of groups

$$\varphi\colon\ \mathrm{PGL}_2(\mathbf{k}(\underline{y})) \xrightarrow{det} \mathbf{k}(\underline{y})^*/(\mathbf{k}(\underline{y})^*)^2 \simeq \bigoplus_P \mathbb{Z}/2 \xrightarrow{pr_I} \bigoplus_I \mathbb{Z}/2$$

where P is the set of prime divisors in $\mathbb{P}^{n-1}_{\mathbf{k}}$ and the isomorphism $\mathbf{k}(\underline{y})^*/(\mathbf{k}(\underline{y})^*)^2 \simeq \bigoplus_P \mathbb{Z}/2$ is given as in the previous section. Here, the subset $I \subset P$ is the set of prime divisors whose support has gonality$\geqslant d_n$. The constant d_n is coming from the boundedness of Fano varieties proven by C. Birkar, but we will not go into it here. The above theorem generalises to decomposable embedded conic bundles [1].

If $\mathbf{k}$ is of characteristic zero, there are many non-equivalent conic bundle structures on birational models of $\mathbb{P}^n_{\mathbf{k}}$, at least as many as the cardinality of $\mathbf{k}$, and the elementary transformations of non-equivalent conic bundle structures do not commute as long as their base-locus have covering gonality large enough [1]. This yields the following result.

Theorem 3.4 ([1]). *Let $n \geqslant 3$ and let $\mathbf{k} \subset \mathbb{C}$ be a subfield. There is a surjective homomorphism*

$$\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}) \longrightarrow \mathop{\ast}_{J} \mathbb{Z}/2,$$

where J is as large as $\mathbf{k}$. In particular, every group generated by a set of involutions of cardinality at most $\mathbf{k}$ is a quotient of $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$. Moreover, the above quotient admits a section, so $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ is a semidirect product with one factor a free product.

In [1] it is also shown that $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$, $n \geqslant 3$, carries the structure of an amalgamated product (with many factors) along the pairwise intersections of the factors. On the other hand, $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is not a non-trivial amalgamated product of two factors [11, Appendix by I. Cornulier], whereas $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{R}})$ has the structure of an amalgamated product of two factors, amalgamated along their intersection [41].

There are constructions of similar homomorphism in [4] that generalise the construction from [30] with generalised Bertini involutions of del Pezzo fibrations above a curve; a del Pezzo fibration $\pi\colon X \longrightarrow B$ over a curve B is a surjective morphism with connected fibres, X and B are normal, the relative Picard rank is $\rho(X/B) = 1$ and the general fibre is a del Pezzo surface. In Theorem 3.5 the authors concentrate on the case where the general fibre is a cubic surface.

Theorem 3.5 ([4]). *For each complex del Pezzo fibration $\pi\colon X \longrightarrow B$ over a curve B with general fibre a del Pezzo surface of degree 3, there is a group homomorphism*

$$\mathrm{Bir}(X) \longrightarrow \mathop{\ast}_{\mathbb{N}} \mathbb{Z}/2$$

whose restriction to the subgroup $\{f \in \mathrm{Bir}(X) \mid \pi \circ f = \pi\}$ is surjective.

The homomorphism in Theorem 3.5 sends Bertini involutions whose base-locus has large gonality onto generators of the free product. [4] also shows following result that is orthogonal to many of the above constructions:

Theorem 3.6 ([4]). *There is surjective homomorphism of groups $\rho\colon \mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}}) \longrightarrow \ast_{\mathbb{N}} \mathbb{Z}/2$ with the following property: denoting by $G \subset \mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$ the subgroup generated by all birational maps $f \in \mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$ such that $\pi \circ f = \pi$ for some rational fibration $\pi\colon \mathbb{P}^3_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ (a rational map whose general fibre is rational), we have $\mathrm{Aut}(\mathbb{P}^3_{\mathbb{C}}) \subsetneq G \subsetneq \ker(\rho)$.*

In particular, the above constructions do not exhaust constructions for quotients of Cremona groups in higher dimension.

In [39], S. Zikas generalises the construction from [30] (see Theorem 2.11) with generalised Bertini involutions of Fano threefolds.

Theorem 3.7 ([39]). *The Cremona group $\mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$ can be written as the free product $\mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}}) \simeq G \ast (\ast_J \mathbb{Z}/2)$, where J is uncountable and where the generators of $\mathbb{Z}/2$ are generalised Bertini involutions of $\mathbb{P}^3_{\mathbb{C}}$.*

An interesting consequence of Theorem 3.7 is that there exist uncountably many automorphism of the group $\mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$ of arbitrary order which are not a composition by inner and field automorphisms [39]. To compare, any automorphism of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is inner, up to a field automorphism [17].

3.2. **Quotients of Cremona groups using Motivic invariants.** Let us introduce a third type of construction of normal subgroups of Cremona groups from [31] using motivic invariants. In principal, the idea is also to construct a surjective non-trivial homomorphism $\mathrm{Bir}(\mathbb{P}^n) \longrightarrow G$ to a group G such that the kernel is non-trivial. However, the construction does not use the Sarkisov program nor geometric group theory, and the group G is a free abelian group.

Let $\mathbf{k}$ be a field and let $\mathrm{Bir}_n/\mathbf{k}$ denote the set of equivalence classes of n-dimensional $\mathbf{k}$-varieties up to birational maps. Let's consider the two-dimensional case. Let $\mathbf{k}$ be a perfect field. Then any birational map f of a smooth projective surface X has a decomposition

$$\begin{array}{ccc} & X & \\ {}^{\pi}\swarrow & & \searrow^{\eta} \\ X & \overset{f}{\dashrightarrow} & X \end{array}$$

where η and π are sequences of blow-ups of closed points. We define

$$\phi(f) := \sum_{i=1}^{n}[p_i] - \sum_{i=1}^{n}[q_i] \in \mathbb{Z}[\mathrm{Bir}_0/\mathbf{k}]$$

where the p_i (resp. q_i) are blown up by η (resp. π) and $[p_i], [q_i]$ are respectively their classes in Bir_0. Note that we can replace the classes of the points by the classes of the exceptional divisors and that we could also define

$$\phi(f) := \sum_{i=1}^{n}[E_i] - \sum_{i=1}^{n}[F_i] \in \mathbb{Z}[\mathrm{Bir}_1/\mathbf{k}]$$

where E_i (resp. F_i) are the curves contracted by η (resp. π) and $[E_i], [F_i]$ are their classes in Bir_1.

Now, let $n \geqslant 3$ and let $f: X \dashrightarrow Y$ be a birational map between n-dimensional varieties. There exists a resolution

$$\begin{array}{ccc} & W & \\ {}^{\pi}\swarrow & & \searrow^{\eta} \\ X & \overset{f}{\dashrightarrow} & Y \end{array}$$

where π and η are birational morphisms. Define

$$\phi(f) := \sum_{i=1}^{n}[E_i] - \sum_{i=1}^{n}[F_i] \in \mathbb{Z}[\mathrm{Bir}_{n-1}/\mathbf{k}]$$

where $E_i \subset X$ (resp. $F_i \subset X$) are the $n-1$ dimensional hypersurfaces contracted by π (resp. η).

Theorem 3.8 ([31] for $n \geqslant 3$ and [32] for $n = 2$)**.**
(1) Let $\mathbf{k}$ be a perfect field. Then the map $\phi\colon \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow \mathbb{Z}[\mathrm{Bir}_0/\mathbf{k}]$ is a group homomorphism.
(2) Let $\mathbf{k}$ be a field and $n \geqslant 3$ and X any n-dimensional variety. Then the map $\phi\colon \mathrm{Bir}(X) \longrightarrow \mathbb{Z}[\mathrm{Bir}_{n-1}/\mathbf{k}]$ is a group homomorphism.

Note that in order to make ϕ a homomorphism, it is necessary to consider the birational classes of the base-loci and not the isomorphism classes (for points, this is the same). Indeed, if f and g are two birational maps of X, $n \geqslant 3$, then g^{-1} may restrict to a birational map (not necessarily an isomorphism) on the base-loci of f.

In [32] the authors show that in fact ϕ is the trivial group homomorphism. More precisely, they show that up to order, η and π blow-up isomorphic points. However, in dimension$\geqslant 3$, the situation is quite different. H.-Y. Lin and E. Shinder show the following statement:

Theorem 3.9 ([31])**.** *The homomorphism $\phi\colon \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}) \longrightarrow \mathbb{Z}[\mathrm{Bir}_{n-1}/\mathbf{k}]$ is non-trivial in the following cases:*

(1) if $\mathbf{k}$ is a number field or is a function field over an algebraically closed field, over a finite field or over a number field and $n \geqslant 3$,

(2) if $\mathbf{k} \subset \mathbb{C}$ *and* $n \geqslant 4$,
(3) if $\mathbf{k}$ *is infinite and* $n \geqslant 5$.

In particular, in these cases, $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ *is not generated by involutions.*

In particular, dimension $n = 2$ is the sporadic case when $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is generated by involutions, if $\mathbf{k}$ is perfect [29].

Once the statement holds in dimension n_0 and there is a birational map f of $\mathbb{P}^{n_0}$ such that $\phi(f)$ is non-zero, we consider the birational map $f \times \mathrm{id}_{\mathbb{P}^m_{\mathbf{k}}}$ of $\mathbb{P}^{n_0} \times \mathbb{P}^m_{\mathbf{k}}$ to prove the statement in dimension $> n_0$. So in each case, it suffices to show the claim in the case of the smallest dimension.

To show (2) they use the following example by K.-W. Lai and B. Hasset in [26]: there is a birational map $f\colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ such that the base-locus of f is a K3 surface and the base-locus of f^{-1} is a K3 surface, but the two K3 surfaces are not birational. For (1) they use a similar type of example where the role of K3 surfaces is replaced by curves; these examples appear in the classification of Fano threefolds of rank two by Mori-Mukai [35, 2) on page 111]. For (3) they use a similar type of example of a birational map of five-dimensional G_2-Grassmannians by A. Ito, M. Miura, S. Okawa, K. Ueda in [25].

While we do not fully understand the kernel of the homomorphism $\phi\colon \mathrm{Bir}(X) \longrightarrow \mathbb{Z}[\mathrm{Bir}_{n-1}/\mathbf{k}]$, there is a beautiful description of the kernel of a group homomorphism that is generated by the images of ϕ.

Let $\mathbf{k}$ be of characteristic zero and let $\mathrm{Var}^{\leqslant m}/\mathbf{k}$ be the set of isomorphism classes of $\mathbf{k}$-varieties of dimension$\leqslant n$ and $K_0(\mathrm{Var}^{\leqslant m}/\mathbf{k})$ the trunctuated Grothendieck ring of varieties of dimension$\leqslant m$; it is the group generated by isomorphism classes $[Y]$ of varieties of dimension$\leqslant n-1$ and the relations are generated by the cut-and-paste relations $[Y] = [U] + [Z]$ for every open set $U \subset Y$ and $Y \backslash U = Z$.

There is an exact sequence

$$\mathrm{Var}^{\leqslant m-1}/\mathbf{k} \xrightarrow{\iota_{m-1}} \mathrm{Var}^{\leqslant m}/\mathbf{k} \xrightarrow{\pi_m} \mathbb{Z}[\mathrm{Bir}_m/\mathbf{k}]$$

where $\pi_m(Y) = 0$ if $\dim Y \leqslant m-1$ and $\pi_m(Y) = [Y_1] + \cdots + [Y_r]$ if $\dim Y = m$ and $Y_1, \ldots, Y_r$ are the irreducible components of Y of dimension m.

If X is an algebraic variety of dimension n, then the group homomorphism $\phi\colon \mathrm{Bir}(X) \longrightarrow \mathbb{Z}[\mathrm{Bir}_{n-1}/\mathbf{k}]$ induces a unique group homomorphism

$$\tilde{\phi}\colon \mathrm{Bir}(X) \longrightarrow K_0(\mathrm{Var}^{\leqslant n-1}/\mathbf{k}),$$

such that if $\iota\colon U \longrightarrow Y$ is the inclusion of an open set, then $\tilde{\phi}(\iota) = [Y \backslash U]$ and such that $\pi_{n-1} \circ \tilde{\phi} = \phi$.

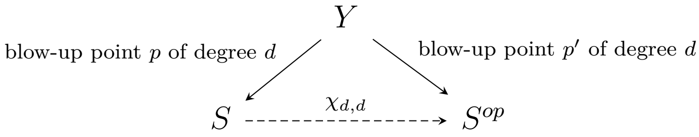
Proposition 3.10 ([31]). *We have* $\ker(\iota_{n-1}) = \sum_{X \in \mathrm{Bir}_n/\mathbf{k}} \tilde{\phi}(\mathrm{Bir}(X))$.

We invite the reader to discover more details in [31].

3.3. **Quotients of Cremona groups using Severi-Brauer surfaces.** Let $\mathbf{k}$ be a field and $\overline{\mathbf{k}}$ its algebraic closure. Châtelet's theorem says that a variety S such that $S_{\overline{\mathbf{k}}} \simeq \mathbb{P}^n_{\overline{\mathbf{k}}}$ has a $\mathbf{k}$-rational point if and only if $S \simeq \mathbb{P}^n_{\mathbf{k}}$. Such a variety S is called *Severi-Brauer variety* and it is called *non-trivial Severi-Brauer variety* if $S(\mathbf{k}) = \varnothing$.

Let us look at a quotient construction for $\mathrm{Bir}(S)$, when S is a non-trivial Brauer-Severi surface, and an application to Cremona groups in higher dimension. We work over a perfect field.

A classical result says that a point p of a non-trivial Severi-Brauer surface S has degree $[\mathbf{k}(p) : \mathbf{k}] = 3e \geqslant 3$, where $\mathbf{k}(p)$ is the residue field of p. The only Sarkisov links starting from S are links of type II whose base-points are of degree 3 or 6



where the surface Y in the diagram is a del Pezzo surface, $d \in \{3, 6\}$ and $\mathbf{k}(p) \simeq \mathbf{k}(p')$. The surface S^{op} is also a non-trivial Severi-Brauer surface that is called *opposite* of S and may or may not be isomorphic to S. It turns out that any point of degree 3 in S is in general position and links $\chi_{3,3}$ as above always exist. However, there are non-trivial Severi-Brauer surfaces with no points of degree 6 in general position [3]. Let P_3 (resp. P_6) be the set of points of degree 3 (resp. 6) in S up to $\mathrm{Aut}(S)$. Looking at relations between Sarkisov links, J. Blanc, J. Schneider and E. Yasinsky prove the following.

Theorem 3.11 ([3])**.** *Let $\mathbf{k}$ be a perfect field and S a non-trivial Severi-Brauer surface. Then $|P_3| \geqslant 2$ and for each point $p \in P_3$ there exists a surjective homomorphism of groups*

$$\mathrm{Bir}(S) \longrightarrow \bigoplus_{P_3 \setminus \{p\}} \mathbb{Z}/3 * \left(\mathop{\ast}_{P_6} \mathbb{Z} \right)$$

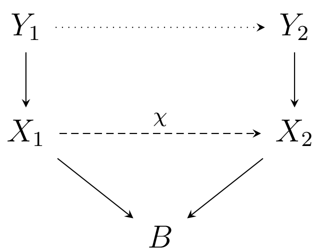
In particular, $\mathrm{Bir}(S)$ *is not simple. Moreover, if $P_6 \neq \varnothing$, then* $\mathrm{Bir}(S)$ *is not generated by elements of finite order.*

The homomorphism is constructed analogously to the constructions of quotients to finite groups in Section 2.2; they first construct a homomorphism

$$\mathrm{BirMori}(S) \longrightarrow \bigoplus_{\mathcal{E}_3 \setminus \{p\}} \mathbb{Z}/3 * \left(\mathop{\ast}_{\mathcal{E}_6} \mathbb{Z} \right),$$

where $\mathcal{E}_3$ and $\mathcal{E}_6$ are equivalence classes of Sarkisov links $\chi_{3,3}$ and $\chi_{6,6}$, respectively. Then they show that after composing with a certain projection, the restriction to $\mathrm{Bir}(S)$ becomes surjective.

Now, let's look at their application for Cremona groups in higher dimension. For $i = 1, 2$, consider a morphisms $X_i \longrightarrow B$ between complex varieties with $\dim B \geqslant 2$ such that X_i has nice singularities and such that the generic fibre $S_i = (X_i)_{\mathbb{C}(B)}$ is a Severi Brauer surface. Notice that the condition $\dim B \geqslant 2$ is important, because if $\dim B = 1$, then $\mathbb{C}(B)$ is a C^1-field, so by Tsen theorem there is a section and hence $S_i \simeq \mathbb{P}^2_{\mathbb{C}(B)}$. So we have $\dim X_i \geqslant 4$. A Sarkisov link χ of type II starting from X_1 is a directed diagram of the form



where $X_i \longrightarrow Y_i$ are divisorial contractions and the dotted arrow is an isomorphism in codimension 1 (see Section 2.2.2). Then χ induces a birational map $S_1 \dashrightarrow S_2$ between the generic fibres [3]. It may be a Sarkisov link and in that case the base-locus of χ is a curve that intersects a general fibre in $d \in \{3, 6\}$ points in general position. With this, J. Blanc, J. Schneider and E. Yasinsky show the following theorem.

Theorem 3.12 ([3])**.**

(1) If $X \longrightarrow B$ is a morphism with $\dim B \geqslant 2$ and $X_{\mathbb{C}(B)}$ a nontrivial Severi-Brauer surface. Then there is a surjective group homomorphism

$$\mathrm{BirMori}(X) \longrightarrow \bigoplus_{M_3} \mathbb{Z}/3 * \left(\mathop{\ast}_{M_6} \mathbb{Z} \right)$$

where M_d, $d \in \{3,6\}$, is a set of particular Sarkisov links.

(2) For each $m \geqslant 4$ there exists an example of such a fibration $X \longrightarrow B$ such that X is rational of dimension m and where the cardinality of M_6 is the cardinality of $\mathbb{C}$.

(3) In particular, if $m \geqslant 4$, there is a surjective homomorphism $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \mathcal{F}(\mathbb{C})$ to the free group $\mathcal{F}(\mathbb{C})$ over the set $\mathbb{C}$, and so $\mathrm{Bir}(\mathbb{P}^m_{\mathbb{C}})$ is not generated by involutions.

As a corollary, they show that for any complex algebraic variety of dimension$\geqslant 4$ there is a surjective homomorphism $\mathrm{Bir}(\mathbb{P}^m_{\mathbb{C}}) \longrightarrow \mathrm{Bir}(X)$ [3].

3.4. **A dream.** We know very little about the kernels of the above group homomorphisms. Birational geometry in dimension$\geqslant 3$ is even harder than it is for threefolds and therefore, we need to view these constructions of homomorphisms as sporadic opportunities that arise because we manage to study some n-folds rather better than others. It would be fantastic to be able to expand these constructions in a way that is not restrained by our knowledge of only certain types of n-folds.

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