A REMARK ON GEISER INVOLUTIONS

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ABSTRACT. We construct a morphism from the plane Cremona group over a perfect field to the free product generated by classes of Geiser involutions, and indicate conditions under which the morphism is non-trivial.

Keywords: Cremona groups, birational geometry

1. Introduction

Geiser and Bertini involutions of $\mathbb{P}^2_{\mathbf{k}}$ are among the most classical birational involutions of the plane. If the base-field $\mathbf{k}$ is perfect but not algebraically closed, they may have one single base-point of degree 7 and 8, respectively, that is, their base-point has 7 (resp. 8) geometric components.

If $\mathbf{k}$ has a Galois extension of degree 8, there is a surjective split morphism of groups $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow \ast_B \mathbb{Z}/2$, where B runs through all representatives of Bertini involutions of $\mathbb{P}^2_{\mathbf{k}}$ over $\mathbf{k}$ up to conjugation by $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}}) \simeq \mathrm{PGL}_3(\mathbf{k})$ with a base-point of degree 8, and B is at least as large as $\mathbf{k}$ [7]. The result is generalised in dimension 3 in [1, 10]. In this article, we show a similar result, but we use Geiser involutions instead of Bertini involutions. However, contrary to [7], we do not give a general estimate on the cardinality of the set of Geiser involutions up to conjugation by automorphisms.

Main Theorem. *Let $\mathbf{k}$ be a perfect field. Then there exists a split surjective homomorphism*

$$\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow \mathop{\ast}_{J(\mathbb{P}^2)} \mathbb{Z}/2 \ast \mathop{\ast}_{X_8 \in I_8} (\ast_{J(X_8)} \mathbb{Z}/2) \ast \mathop{\ast}_{X_6 \in I_6} (\ast_{J(X_6)} \mathbb{Z}/2) \mathop{\ast}_{X_5 \in I_5} (\ast_{J(X_5)} \mathbb{Z}/2)$$

where I_d is the set of isomorphism classes of rational del Pezzo surfaces X_d of degree d with $\mathrm{NS}(X_d) \simeq \mathbb{Z}$ and $J(X_d)$ is the set of Geiser involutions of X_d up to conjugation by $\mathrm{Aut}(X_d)$. The sets $J(\mathbb{P}^2)$ and $J(X_d)$ are non-empty in the following cases:

- *$J(\mathbb{P}^2)$: $\mathbf{k}$ has a Galois extension $L/\mathbf{k}$ with a $\mathrm{Gal}(L/\mathbf{k})$-orbit of cardinality 7.*
- *$d = 8$: X_8 exists, $|\mathbf{k}| \neq 7$ and $\mathbf{k}$ has a Galois extension $L/\mathbf{k}$ with a $\mathrm{Gal}(L/\mathbf{k})$-orbit of cardinality 6.*
- *$d = 6$: X_d exists, $|\mathbf{k}| \neq 4$ and has a Galois extension $L/\mathbf{k}$ with a $\mathrm{Gal}(L/\mathbf{k})$-orbit of cardinality 4.*
- *$d = 5$: X_d exists, $|\mathbf{k}| \neq 4$, $\mathbf{k}$ has a Galois extension $L/\mathbf{k}$ with a $\mathrm{Gal}(L/\mathbf{k})$-orbit of cardinality 3 and the base-point of a Sarkisov links $\mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathcal{S}_5$ is on a nodal cubic given by $xyz = c(x^3 - z^3)$ for some $c \in \mathbf{k}^*$.*

Moreover, if $\mathbf{k}$ is infinite and $J(\mathbb{P}^2_{\mathbf{k}})$ (resp. $J(X_d)$) is non-empty, then $J(\mathbb{P}^2_{\mathbf{k}})$ (resp. $J(X_d)$) is at least as large as $\mathbf{k}$.

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The homomorphism in the Main Theorem is trivial if $\mathbf{k}$ is algebraically closed. The cardinality of I_d and $J(X_d)$ depends on the field: for instance $|I_8| = 1$ if $\mathbf{k} = \mathbb{R}$ and $|I_8|$ is infinite if $\mathbf{k} = \mathbb{Q}$. If $\mathbf{k}$ is a finite field, then there are only a finite number of points in X_d that can be the base-point of a Geiser involution and then $J(X_d)$ is finite.

In [9], J. Schneider constructs several homomorphism from $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ to a product of $\mathbb{Z}/2$, where $\mathbb{Z}/2$ are generated by images of birational maps preserving a rational conic fibration. In particular, she obtains for any finite field $\mathbf{k}$ a surjective morphism of groups $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$, where the $\mathbb{Z}/2$ are respectively generated by images of birational maps preserving the pencil of lines through a rational point and the pencils of conics passing through two points of degree 2 or through a point of degree 4.

To show the Main Theorem, we follow the idea from [7, 9]. We embed $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ into the groupoid $\mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}})$ of birational maps between rational Mori fibre spaces over $\mathbf{k}$ of dimension 2, which is generated by Sarkisov links and isomorphisms of Mori fibre spaces [2, 4]. According to [7], any relation in $\mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}})$ is generated by trivial relations and *elementary relations* between Sarkisov links. A complete list of elementary relations can be found in [6, 11]. Geiser involutions with only one base-point are particular Sarkisov links. We write down all types of elementary relations involving a Geiser involution (see §2.2), which then allows us to construct a homomorphism

$$\mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow \mathop{\ast}_{J(\mathbb{P}^2)} \mathbb{Z}/2 * \mathop{\ast}_{X_8\in I_8} (*_{J(X_8)}\mathbb{Z}/2) * \mathop{\ast}_{X_6\in I_6} (*_{J(X_6)}\mathbb{Z}/2) \mathop{\ast}_{X_5\in I_5} (*_{J_5}\mathbb{Z}/2).$$

Its restriction to $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is the homomorphism in the Main Theorem.

In [3], V.A. Iskovskikh provides a generating set of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$, where $\mathbf{k}$ is a perfect field, and in [5], V.A. Iskovskikh, F.K. Kabdykairov and S.L. Tregub present a set of generating relations for the generating set from [3]. The list is very long, and one reason for this is that the authors insist on seeing the relations as relations between maps $\mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$. However, the generating set derives from the Sarkisov program (even though the construction of the Sarkisov program in [2, 4] was published after [3], it is already visible in [3]) and the generating relations become easier when we write them as relations between Sarkisov links. Checking the list of generating relations in [5], we see that Geiser involutions of $\mathbb{P}^2_{\mathbf{k}}$ (written χ_7) appear only in relation [5, §2.3(ii)], Geiser involutions of X_8 (written $\chi_{2,6}$) appear only in relations [5, §2.3(ii), §2.4(i), §2.7(ii)], Geiser involutions of X_6 (written $\chi_{2,3,4}$) appear only in [5, §2.3(ii), §2.5(iv),(v), §2.6(iv), §2.7(x),(xiv)] and Geiser involutions of X_5 (written $\chi_{5,3}$) appear only in [5, §2.3(ii), §2.4(iii),(iv)]. These relations decompose into conjugates of *elementary relations*, which is why there are more relations in [5] involving Geiser involutions than listed here in §2.2.

From now on, $\mathbf{k}$ is a perfect field, $\overline{\mathbf{k}}$ its algebraic closure and if $L/\mathbf{k}$ is a Galois extension, then $\mathrm{Gal}(L/\mathbf{k})$ denotes the Galois group of L over $\mathbf{k}$.

2. Relations in the Cremona group involving a Geiser involution

2.1. **Sarkisov links.** For a projective variety X, we have $\mathbf{k}[X_{\overline{\mathbf{k}}}]^* = (\overline{\mathbf{k}})^*$. Hence, if $X(\mathbf{k}) \neq \varnothing$, we have $\mathrm{Pic}(X_{\mathbf{k}}) = \mathrm{Pic}(X_{\overline{\mathbf{k}}})^{\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})}$ [8, Lemma 6.3(iii)]. This holds in particular if X is $\mathbf{k}$-rational, because then it has a $\mathbf{k}$-rational point by the Lang-Nishimura theorem. In particular, we have $\mathrm{NS}(X) = \mathrm{NS}(X_{\overline{\mathbf{k}}})^{\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})}$. For a surjective morphism $\pi\colon X \longrightarrow B$, we denote by

$\mathrm{NS}(X/B)$ the quotient of $\mathrm{NS}(X)$ by the subspace of divisors whose intersection with the contracted curves is zero. We denote by $\rho(X/B)$ the rank of $\mathrm{NS}(X/B)$. We call the *splitting field* of a point p the smallest Galois extension L of $\mathbf{k}$ such that all geometric components of p are defined over L. Points in a del Pezzo surface are *in general position* (with each other) if their blow-up is again a del Pezzo surface.

A *rank r fibration* in dimension 2 is a surjective morphism $\pi\colon X \longrightarrow B$ from a smooth surface X with $X(\mathbf{k}) \neq \varnothing$ to a curve or point B such that η has connected fibres, $-K_X$ is π-ample and $r = \rho(X/B) \geqslant 1$. We denote a rank r fibration also by X/B.

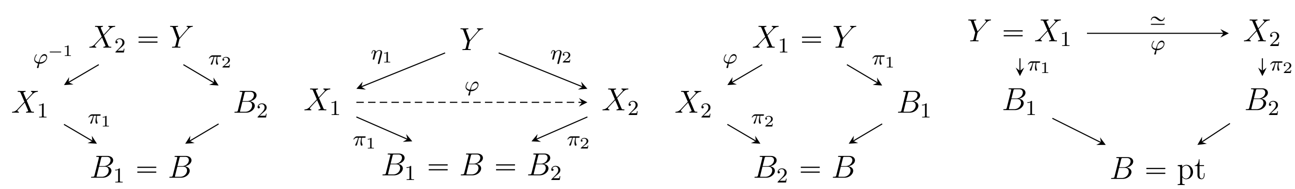
A rank 1 fibration X/B is a Mori fibre space, so if B is a point, then X is a del Pezzo surface, and if B is a curve, then X/B is a conic fibration.

If $r \geqslant 2$, we say that a rank r fibration $\pi\colon X \longrightarrow B$ dominates a rank s fibration $\pi'\colon X' \longrightarrow B'$ if there exists a birational morphism $X \longrightarrow X'$ and a morphism $B' \longrightarrow B$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \rightarrow & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \leftarrow & B' \end{array}$$

In particular, we have $r \geqslant s$. We also say that X/B *factorises through* X'/B'.

For a rank 2 fibration Y/B, there exist at most two extremal rays in $\mathrm{NS}(Y/B)$, so Y/B dominates at most two rank 1 fibrations $\pi\colon X_1 \longrightarrow B_1$ and $\pi\colon X_2 \longrightarrow B_2$. The two contractions induce the following four types of diagrams, called *Sarkisov diagrams*,



type I type II type III type IV

where all morphisms $Y \longrightarrow X_i$ are birational morphisms and all morphisms that are not isomorphisms have relative Picard number 1. The birational maps $\varphi\colon X_1 \dashrightarrow X_2$ are called *Sarkisov links*. The inverse of a Sarkisov link of type III is a Sarkisov link of type I.

Here is an interpretation for rank 3 fibrations.

Proposition-Definition 2.1 ([7, Proposition 2.6])**.** *Let T/B be a rank 3 fibration over $\mathbf{k}$. Then there exist only a finite number of rank 2 fibrations dominated by T/B. In particular, there exist only finitely many Sarkisov links $\chi_1, \ldots, \chi_m$ that are dominated by T and they fit into a relation $\chi_m \circ \cdots \circ \chi_1 = \mathrm{id}$. We call such a relation* elementary relation.

For a surface X over $\mathbf{k}$, we denote by $\mathrm{BirMori}(X)$ the groupoid of birational maps between Mori fibre spaces birational to X.

Theorem 2.2 ([2, Appendix],[4, Theorem 2.5],[7, Theorem 3.1(1)])**.** *Let X/B a Mori fibre space over $\mathbf{k}$ of dimension 2. Then the groupoid $\mathrm{BirMori}(X)$ is generated by Sarkisov links and isomorphisms.*

Theorem 2.3 ([7, Theorem 3.1(2)])**.** *Let X/B a Mori fibre space over $\mathbf{k}$ of dimension 2. Then any relation of $\mathrm{BirMori}(X)$ is generated by trivial relations and elementary relations.*

2.2. **Relations involving Geiser involutions.** Let X be a del Pezzo surface over $\mathbf{k}$ with $X(\mathbf{k}) \neq \varnothing$, $\rho(X) = 1$ and $K_X^2 \geqslant 3$. Then $X \longrightarrow *$ is a rank 1 fibration. Suppose that there exists a birational morphism $\eta\colon Y \longrightarrow X$ such that Y is a del Pezzo surface with $K_Y^2 = 2$ and $\rho(Y) = 2$, that is, η contracts a $\mathbf{k}$-irreducible curve with $n = K_Y^2 - K_X^2$ geometric components onto a point p whose n geometric components are in general position. Then Y dominates a Sarkisov link $\chi\colon X \dashrightarrow X$ we call *Geiser link*. Geometrically, it can be seen as follows: the linear sytem $|-K_Y|$ induces a double cover $Y \longrightarrow \mathbb{P}^2_{\mathbf{k}}$ ramified above a smooth plane quartic curve. The Galois involution of the double cover induces a birational involution $\gamma\colon X \dashrightarrow X$, called *Geiser involution*. Its base-locus is the point p and there exist $\alpha, \beta \in \operatorname{Aut}(X)$ such that $\chi = \beta \circ \gamma \circ \alpha$.

A complete list of all elementary relations in $\operatorname{BirMori}(\mathbb{P}^2_{\mathbf{k}})$ can be found in [6, 11]. We now present a list of elementary relations involving a Geiser link and then show that the list is exhaustive. It is not clear that all relations below actually exist over a given field $\mathbf{k}$, because the points implicated in the blow-ups may not exist in sufficiently general position.

In what follows, we indicate the geometric components of the contracted curves, and T always denotes a del Pezzo surface. The dashed lines in the diagrams represent Sarkisov links, and the letter next to an arrow indicates the curve contracted by the corresponding birational morphism.

2.2.1. *Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a rational point and a point of degree* 7. Denote respectively by $E \subset T$ and $E_1', \dots, E_7' \subset T_{\overline{\mathbf{k}}}$ the geometric components of their exceptional divisors. Let $L \subset T$ be the pullback of a general line in $\mathbb{P}^2$. The only $\operatorname{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves on $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 8$ with pairwise disjoint members are the following:

$E, \quad E_1', \dots, E_7', \quad \ell_i := L - E - E_i'$

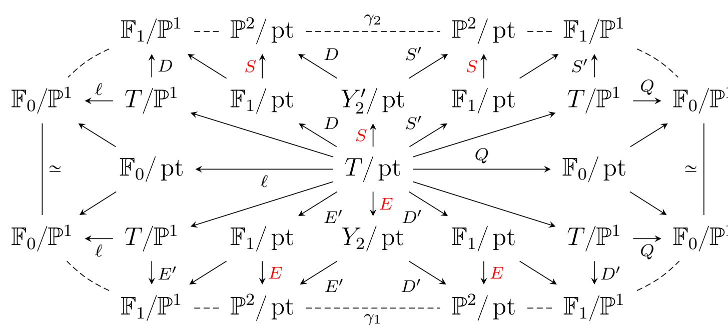
$D_{i_7} := 3L - 2E - E_{i_1}' - \cdots - E_{i_6}'$

$D_{i_1}' := 3L - 2E_{i_1}' - E_{i_2}' - \cdots - E_{i_7}'$

$Q_{i_1} := 5L - E - E_{i_1}' - 2E_{i_2}' - \cdots - 2E_{i_7}'$

$S := 6L - 3E - 2E_1' - \cdots - 2E_7',$

$S_{i_1}' := 6L - 3E_{i_1}' - 2E_{i_2}' - \cdots - 2E_{i_7}' - 2E.$



We have

$$E \cdot E_i' = E \cdot D_i' = D_i \cdot S = S \cdot S_i' = 0, \quad E_i' \cdot \ell_j = \ell_i \cdot D_i = D_i' \cdot Q_j = Q_i \cdot S_j' = \delta_{ij}, \quad E_i' \cdot D_j = 1 - \delta_{ij}$$

for all i, j, where δ_{ij} is the Kronecker delta. Completing the contraction diagram, we obtain the commutative diagram above, where by E' we mean $E_1' + \cdots + E_7'$ and so forth. The birational maps $\gamma_1, \gamma_2\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ are Geiser links.

2.2.2. *Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a point of degree* 2 *and a point of degree* 6. Denote respectively by E_1, E_2 and $E_1', \dots, E_6'$ the geometric components of their exceptional divisors. Let L be the pullback of a general line in $\mathbb{P}^2$. The only $\operatorname{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 8$ with pairwise disjoint members are the following:

$E_1, E_2, \quad E'_1, \dots, E'_6, \quad \ell := L - E_1 - E_2$

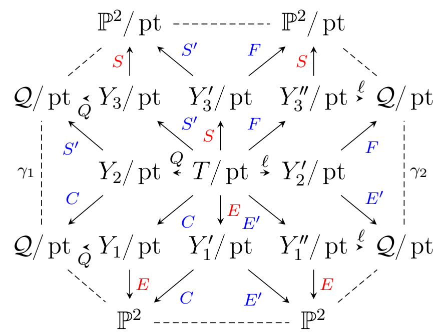
$C_{i_6} = 2L - E'_{i_1} - \cdots E'_{i_5}$

$F_{i_1} := 4L - 2E_1 - 2E_2 - 2E'_{i_1} - E'_{i_2} - \cdots - E'_{i_6}$

$Q := 5L - E_1 - E_2 - 2E'_1 - \cdots - 2E'_6$

$S_i := 6L - 3E_i - 2E_{3-i} - 2E'_1 - \cdots - 2E'_6$

$S'_{i_1} := 6L - 3E'_{i_1} - 2E_1 - 2E_2 - 2E'_{i_2} - \cdots - 2E'_{i_6}.$



We have

$$E_i \cdot E'_j = E_i \cdot C_j = E'_i \cdot \ell = \ell \cdot F_i = C_i \cdot Q = F_i \cdot S_j = Q \cdot S'_i = S_i \cdot S'_j = 0, \quad E'_i \cdot C_j = \delta_j, \quad F_i \cdot S'_j = 1 - \delta_{ij}$$

for all i, j. This yields again all possible contractions from T to a rank 2 fibration and we obtain the commutative diagram above. Two quadric surfaces in the diagram joined by a Sarkisov link are joined by a Geiser link and are hence isomorphic, and it follows that all quadric surfaces in the diagram are isomorphic.

2.2.3. *Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a point of degree* 3 *and a point of degree* 5. Denote respectively by E_1, E_2, E_3 and $E'_1, \dots, E'_5$ the geometric components of their exceptional divisors. Let $L \subset T$ be the pullback of a general line in $\mathbb{P}^2$. Among the 240 (-1)-curves on $T_{\overline{\mathbf{k}}}$, the only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of cardinality $\leqslant 8$ are the following:

$E_1, E_2, E_3, \quad E'_1, \dots, E'_5,$

$\ell_i := L - E_1 - E_2 - E_3 - E'_i$

$C := 2L - E'_1 - \cdots - E'_5$

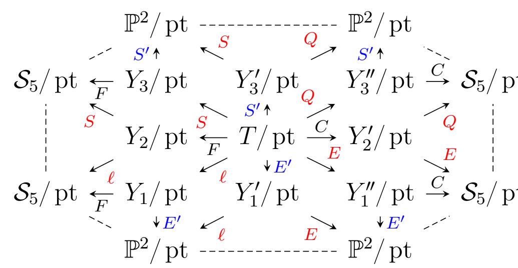
$D_{i_1} := 3L - 2E_{i_1} - E_{i_2} - E'_1 - \cdots - E'_5$

$F := 4L - 2E_1 - 2E_2 - 2E_3 - E'_1 - \cdots - E'_5$

$Q_{i_3} := 5L - E_{i_1} - E_{i_2} - 2E_{i_3} - 2E'_1 - \cdots - 2E'_5$

$S_{i_1} := 6L - 3E_{i_1} - 2E_{i_2} - 2E_{i_3} - 2E'_1 - \cdots - 2E'_5$

$S'_{i_1} := 6L - 3E'_{i_1} - 2E'_{i_2} - \cdots - 2E'_{i_5} - 2E_1 - 2E_2 - 2E_3$



The orbit $D_1, \cdots, D_6$ has intersecting members, so it cannot be contracted from T. We have

$$E_i \cdot E'_j = E_i \cdot C = E'_j \cdot \ell_i = C \cdot Q_i = F \cdot S_i = F \cdot \ell = S_i \cdot S'_j = Q \cdot S'_j = 0$$

for any $i = 1, 2, 3,\ j = 1, \dots, 5$. This yields the commutative diagram above, where $\mathcal{S}_5$ is a del Pezzo surface of degree 5. Two del Pezzo surfaces of degree 5 in the diagram joined by a Sarkisov link are joined by a Geiser link, hence the two surfaces are isomorphic. It follows that all del Pezzo surfaces of degree 5 in the diagram are isomorphic.

2.2.4. *Let $\mathcal{Q} \subset \mathbb{P}^3$ be a rational quadric surface with $\rho(\mathcal{Q}) = 1$ and $T \longrightarrow \mathcal{Q}$ the blow-up of a point of degree* 3 *and a point of degree* 4. Denote respectively by E_1, E_2, E_3 and $E'_1, \dots, E'_4$ the geometric components of their exceptional divisors. Let $F \subset T$ be the pullback of the generator of $\mathrm{NS}(\mathcal{Q})$. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves on $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 7$ and pairwise disjoint members are the following, where the number of geometric components is indicated in parenthesis.

$E_1, E_2, E_3, \quad E'_1, \dots, E'_4,$

$\ell := F - E_1 - E_2 - E_3$ (1)

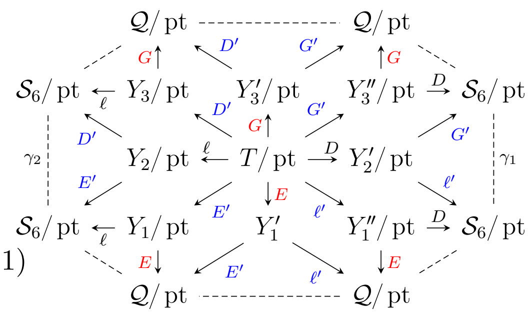
$\ell'_{i_4} := F - E'_{i_1} - \cdots - E'_{i_3}$ (1)

$D := 3F - 2E'_1 - \cdots - 2E'_4 - E_1 - E_2 - E_3$ (1)

$D'_{i_1} := 3F - 2E_1 - 2E_2 - 2E_3 - 2E'_{i_1} - E'_{i_2} - \cdots - E'_{i_4}$ (1)

$G_{i_1} := 4F - 3E_{i_1} - 2E_{i_2} - 2E_{i_3} - 2E'_1 - \cdots - 2E'_4$ (1)

$G'_i := 4F - 3E'_{i_1} - 2E_{i_2} - \cdots - 2E_{i_4} - 2E_1 - 2E_2 - 2E_3$ (1)



We have

$$E_i \cdot E'_j = E_i \cdot \ell'_j = E'_i \cdot \ell = \ell \cdot D'_i = \ell'_i \cdot D = D'_i \cdot G_j = D \cdot G'_i = G_i \cdot G'_j = 0, \quad D'_i \cdot G'_j = 1 - \delta_{ij}$$

for all i, j. All other pairs of orbits have no trivial intersections. Completing the commutative diagram yields the diagram above. Two del Pezzo surfaces of degree 6 in the diagram joined by a Sarkisov link are joined by a Geiser link and are hence isomorphic. It follows that all del Pezzo surfaces of degree 6 in the diagram are isomorphic.

2.2.5. *The list is exhaustive.*

Lemma 2.4. *Any elementary relation between Sarkisov links involving a Geiser link between rational del Pezzo surfaces over* $\mathbf{k}$ *is one of the relations in §2.2.1, §2.2.3, §2.2.2, §2.2.4.*

Proof. The rational del Pezzo surfaces over $\mathbf{k}$ with Picard rank 1 are $\mathbb{P}^2$, quadric surfaces $\mathcal{Q} \subset \mathbb{P}^3$ with a rational point and $\rho(\mathcal{Q}) = 1$, and del Pezzo surfaces $\mathcal{S}_5, \mathcal{S}_6$ of degree 6 or 5 with a rational point and $\rho(\mathcal{S}_i) = 1$, $i = 5, 6$. This follows, for instance, from the classification of Sarkisov links in [4, Theorem 2.6]. By definition, any elementary relation of Sarkisov links is dominated by a rank 3 fibration T/B. If this elementary relation involves a Geiser link, then T is a del Pezzo surface of degree $K_T^2 = 1$ and B is a point, and there is a birational morphism $T \longrightarrow X$, where X is one of the rank 1 fibrations $\mathbb{P}^2/*$, $\mathcal{Q}/*$, $\mathcal{S}_6/*$ or $\mathcal{S}_5/*$ In particular, we have the following options (X, d_1, d_2) for the degrees d_1, d_2 of the two points blown up by $T \longrightarrow X$:

$$(\mathbb{P}^2_{\mathbf{k}}; 1, 7), (\mathbb{P}^2_{\mathbf{k}}; 2, 6), (\mathbb{P}^2_{\mathbf{k}}; 3, 5), (\mathbb{P}^2_{\mathbf{k}}; 4, 4) \qquad (\mathcal{S}_6; 1, 4), (\mathcal{S}_6; 2, 3)$$
$$(\mathcal{Q}; 1, 6), (\mathcal{Q}; 2, 5), (\mathcal{Q}, 3, 4) \qquad (\mathcal{S}_5; 1, 3), (\mathcal{S}_5, 2, 2).$$

The options $(\mathcal{Q}; 1, 6), (\mathcal{S}_6; 1, 4), (\mathcal{S}_5; 1, 3), (\mathcal{S}_5, 2, 2)$ appear respectively in the diagrams given by $(\mathbb{P}^2_{\mathbf{k}}; 2, 6), (\mathcal{Q}, 3, 4), (\mathbb{P}^2_{\mathbf{k}}; 3, 5), (\mathcal{Q}; 2, 5)$. One checks that $(\mathbb{P}^2_{\mathbf{k}}; 4, 4), (\mathcal{Q}; 2, 5), (\mathcal{S}_6; 2, 3)$ lead to diagrams not involving any Geiser links, see for instance [6, 11]. The remaining cases $(\mathbb{P}^2_{\mathbf{k}}; 1, 7)$, $(\mathbb{P}^2_{\mathbf{k}}; 2, 6)$, $(\mathbb{P}^2_{\mathbf{k}}; 3, 5)$ and $(\mathcal{Q}, 3, 4)$ are correspond, respectively, to the relations in §2.2.1, §2.2.2 and §2.2.3, §2.2.4. □

We now show that two Geiser involutions appearing in the same elementary relation are the same up to composition with automorphisms.

Lemma 2.5.

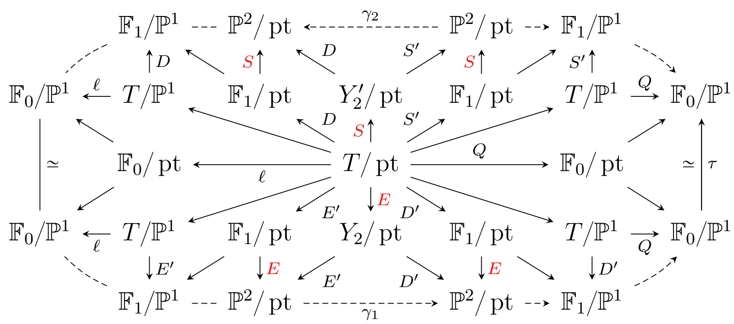
(1) Let $\gamma_1, \gamma_2 \colon \mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ *be two Geiser involutions appearing in a non-empty relation given in §2.2.1. Then* $\gamma_2 = \alpha\gamma_1\alpha^{-1}$ *for some* $\alpha \in \mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$.

(2) Let $\gamma_1, \gamma_2 \colon \mathcal{Q} \dashrightarrow \mathcal{Q}$ *be two Geiser involutions appearing in a non-empty relation given in §2.2.2. Then* $\gamma_2 = \alpha\gamma_1\alpha^{-1}$ *for some* $\alpha \in \mathrm{Aut}(\mathcal{Q})$.

(3) Let $\gamma_1, \gamma_2 \colon \mathcal{S}_5 \dashrightarrow \mathcal{S}_5$ *be two Geiser involutions appearing in a non-empty relation given in §2.2.3. Then* $\gamma_2 = \alpha\gamma_1\alpha^{-1}$ *for some* $\alpha \in \mathrm{Aut}(\mathcal{S}_5)$.

(4) Let $\gamma_1, \gamma_2 \colon \mathcal{S}_6 \dashrightarrow \mathcal{S}_6$ *be two Geiser involutions appearing in a non-empty relation given in §2.2.4. Then* $\gamma_2 = \alpha\gamma_1\alpha^{-1}$ *for some* $\alpha \in \mathrm{Aut}(\mathcal{S}_6)$.

Proof. The surface T is a del Pezzo surface of degree 1 and its Bertini involution acts on the (-1)-curves of $T_{\overline{\mathbf{k}}}$ and hence on the relation diagram. It does not preserve any (-1)-



curve in $T_{\overline{\mathbf{k}}}$, so it acts as rotation. (It is called *central symmetry* of the relation diagram in [6].) Then the birational map $\beta \colon \mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ obtained when tracing the arrows denoted by E', E, S, S' is a Bertini involution of $\mathbb{P}^2_{\mathbf{k}}$ up to automorphisms of $\mathbb{P}^2_{\mathbf{k}}$, that is, $\phi = \alpha_2\beta\alpha_1$ for some $\alpha_1, \alpha_2 \in \mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$. The base-point p_1 of γ_1 is a base-point of φ and the base-point p_2 of γ_2 is a base-point of φ^{-1}. Since β is an involution, we have $\alpha_1(p_1) = \alpha_2^{-1}(p_2)$. Therefore, $\gamma_2 = (\alpha_2\alpha_1)\gamma_1(\alpha_2\alpha_1)^{-1}$.

The argument is the same for the remaining three cases. □

3. The homomorphism from the Cremona group to the free product

Recall that I_d denotes the set of isomorphism classes of rational del Pezzo surfaces X_d of degree d with $\rho(X_d) = 1$, and $J(X_d)$ denotes the set of Geiser involutions of X_d up to conjugation by $\mathrm{Aut}(X_d)$.

Proposition 3.1. *There exists a homomorphism*

$$\Psi \colon \mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow \mathop{\ast}_{J_9(\mathbb{P}^2)} \mathbb{Z}/2 * \left(\mathop{\ast}_{X_8 \in I_8} (*_{J(X_8)}\mathbb{Z}/2)\right) * \left(\mathop{\ast}_{X_6 \in I_6} (*_{J(X_6)}\mathbb{Z}/2)\right) * \left(\mathop{\ast}_{X_5 \in I_5} (*_{J(X_5)}\mathbb{Z}/2)\right)$$

that sends Geiser involutions onto the generator indexed by the corresponding class. Moreover, $\Psi \colon \mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow \mathrm{Im}(\Psi)$ *is split.*

Proof. We define the homomorphism as follows. We send all isomorphisms of Mori fibre spaces onto zero, as well as any Sarkisov link that is not a Geiser link. If a del Pezzo surface X_d of degree d with $\rho(X_d) = 1$ is rational, then $d \geqslant 5$. This follows for instance from [4, Theorem 2.6]. Moreover, $\rho(X_7) \geqslant 2$, so we have $d \in \{9, 8, 6, 5\}$, and $X_d \simeq \mathbb{P}^2$ by Châtelet's theorem. We send a Geiser link of X_d onto the generator indexed by $J(X_d)$. Then Ψ is well-defined and it is a morphism of groupoids: by Lemma 2.4 any relation among Sarkisov links including a Geiser link appears among relations 2.2.1–2.2.4. If a relation including a Geiser link of X_d is empty but Geiser links of X_d do exist, then there is nothing further to check. If a relation including a Geiser link is non-empty, then Lemma 4.4 implies that the two Geiser links in the relation are in the same class in $J(X_d)$.

If the image of Ψ is non-trivial, it is the free product generated by the non-trivial images of Geiser links. Let $(e_i)_i \in \mathrm{Im}(\Psi)$ be an element with only one non-zero entry, and let this entry be indexed by i_0. Let γ_{i_0} be a Geiser involution in $\Psi^{-1}((e_i)_i)$. We set $\theta((e_i)_i) = \gamma_{i_0}$. Then $\theta\colon \mathrm{Im}(\Psi) \longrightarrow \mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}})$ is a split of Ψ. □

We prove the Main theorem with the counting statements from the next section.

Proof of Main Theorem. The split homomorphism is from Proposition 3.1. For $d = 9$, the claim follows from Proposition 4.4. For $d = 8$, the claim follows from Proposition 4.9. For $d = 6$, the claim follows from Proposition 4.13. For $d = 5$ the claim follows from Proposition 4.16. □

4. Counting Geiser involutions up to conjugation by automorphisms

We want to show that $J(\mathbb{P}^2_{\mathbf{k}})$ and $J(X_d)$ for $d = 8, 6, 5$ are non-empty and that they are at least as large as $\mathbf{k}$ under certain conditions. In [7] it is shown that if $\mathbf{k}$ is a field with a Galois-extension $L/\mathbf{k}$ of degree 8, then there are at least as many $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$-orbits of points of degree 8 in general position in $\mathbb{P}^2_{\mathbf{k}}$ as the cardinality of $\mathbf{k}$. We use the idea of their proof.

Let $C_P \subset \mathbb{P}^2_{\mathbf{k}}$ be the cubic curve given by $xyz = P(x, z)$, where $P(x, z) = c(x^3 - z^3)$, $c \in \mathbf{k}^*$. It is nodal in the point $[0 : 1 : 0]$, and the nodal point is the only intersection of C_P and the line given by $z = 0$. The two tangent directions at the node are given by $xz = 0$.

We use the following statement often, so we repeat it here:

Lemma 4.1 ([7, Lemma 4.7]). *Consider a collection of six points a_i in $\overline{\mathbf{k}}$. Then:*

1. *The points $(a_i, \frac{P(a_i,1)}{a_i}) \in C_P$, $i = 1, 2, 3$, are on a same line if and only if $a_1a_2a_3 = 1$.*
2. *The points $(a_i, \frac{P(a_i,1)}{a_i}) \in C_P$, $i = 1, \ldots, 6$ are on a same conic if and only if $a_1 \cdots a_6 = 1$.*

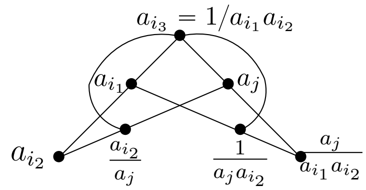
Remark 4.2. Let p be a prime number. Let $L/\mathbf{k}$ be a Galois extension such that $\mathrm{Gal}(L/\mathbf{k})$ has an orbit $a_1, \ldots, a_p$. Then $\mathbf{k}(a_1, \ldots, a_p) \subset L$ is a normal subfield and $\mathrm{Gal}(\mathbf{k}(a_1, \ldots, a_p)/\mathbf{k})$ is a subgroup of sym_p that acts transitively on $a_1, \ldots, a_p$, so its order is divible by p but not by p^2. By Sylow's theorem, it contains a cyclic subgroup H of order p. We can extend a generator of H to L, so $\mathrm{Gal}(L/\mathbf{k})$ contains a cyclic subgroup of order p acting non-trivially on $\{a_1, \ldots, a_p\}$. We will use this for $p = 3, 5, 7$.

4.1. The set of Geiser involutions of $\mathbb{P}^2_{\mathbf{k}}$ up to automorphisms.

Lemma 4.3. *Suppose that $\mathbf{k}$ admits a Galois extension $L/\mathbf{k}$ such that $\mathrm{Gal}(L/\mathbf{k})$ has an orbit $a_1, \ldots, a_7 \in L$. Then the $p_i = (a_i, \frac{P(a_i,1)}{a_i})$ are in general position in $\mathbb{P}^2_L$. Moreover, for any smooth rational point $q = (b, \frac{P(b,1)}{b})$ of C_P, the points $q, p_1, \ldots, p_7$ are either in general position or $a_1 \cdots a_7 b^2 = 1$.*

Proof. By Remark 4.2, there is a cyclic subgroup $H \subset \mathrm{Gal}(L/\mathbf{k})$ of order 7 acting non-trivially on $\{a_1, \ldots, a_7\}$. Suppose that a line D in $\mathbb{P}^2_L$ contains q and two of the p_i. Then for each $\sigma \in H$, the line $\sigma(D)$ contains q and two of the p_i. But there are only seven p_i, so D must contain at least three of the p_i, which contradicts $D \cdot C_P = 3$.

Suppose that three of p_i are on a line D. Since the p_i's lie on the singular cubic C_P, D contains exactly three of the p_i, say $p_{i_1}, p_{i_2}, p_{i_3}$. Then $a_{i_1}a_{i_2}a_{i_3} = 1$ by Lemma 4.1 and for any $\sigma \in H$, the line $\sigma(D)$ contains exactly three of the p_i as well. If $\sigma(D)$ and D do not intersect in one of the p_i, then $\sigma^2(L)$ intersects D in one of the p_i. We replace σ by σ^2 and thusly assume that $\sigma(D), D$ intersect in some p_i. Suppose that $(a_{i_1}, \frac{P(a_{i_1},1)}{a_{i_1}}), (a_{i_2}, \frac{P(a_{i_2},1)}{a_{i_2}}) \in D$ and $(a_j, \frac{P(d,1)}{a_j}) \in \sigma(D)\backslash D$. Tracing the orbit of D under H, we obtain the following picture



where we cannot include anymore lines in the H-orbit of D. But there should be seven lines in the picture. We conclude that no three of the p_i are collinear.

Suppose that a conic D contains q and five of the p_i. Since H is cyclic of order seven, the intersection $\sigma(D) \cap D$ contains q and four of the p_i for any $\sigma \in H$. Thus $\sigma(D) = D$ for any $\sigma \in H$, which means that D contains all seven p_i, which contradicts $D \cdot C_P = 6$.

Suppose that a conic D contains six of the p_i. Then for any $\sigma \in H$, the intersection $D \cap \sigma(D)$ contains five of the p_i and hence $\sigma(D) = D$. Again, this contradicts $D \cdot C_P = 6$.

Suppose that D is a cubic singular at p_1 and containing $p_2, \dots, p_7, q$. For any $\sigma \in H$ we have $\sigma(D) \neq D$ and $\sigma(D) \cdot D = 2 \cdot 1 + 2 \cdot 1 + 5 + 1 = 10$, impossible. Finally, let D be a cubic singular at q and containing $p_1, \dots, p_7$. It does not intersect C_P at $[0 : 1 : 0]$, so it is given by an equation

$$A_1x^3 + y^3 + A_3z^3 + A_4x^2y + A_5x^2z + A_6y^2z + A_7yz^2 + A_8xz^2 + A_9xy^2 + A_{10}xyz = 0$$

for $A_1, \dots, A_{10} \in \mathbf{k}$. Setting $z = 1$ and $y = \frac{P(x,1)}{x}$ and multiplying by x^3 yields

$$A_1x^6 + P(x,1)^3 + A_3x^3 + A_4x^4P(x,1) + A_5x^5 + A_6xP(x,1)^2 + A_7x^2P(x,1) + A_8x^4$$
$$+ A_9x^2P(x,1)^2 + A_{10}x^3P(x,1) = c_0^3\prod_{i=1}^{7}(x - a_i)(x - b)^2$$

The constant term of the left-hand side is $-c_0^3$, so $a_1 \cdots a_7 b^2 = 1$. □

Proposition 4.4. *Suppose that* $\mathbf{k}$ *admits a Galois-extension* $L/\mathbf{k}$ *such that* $\mathrm{Gal}(L/\mathbf{k})$ *has an orbit in* L *of cardinality* 7*, and let* G *be the set of Geiser involutions of* $\mathbb{P}^2_{\mathbf{k}}$ *up to conjugation with* $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$*. Then* G *is non-empty. If* $\mathbf{k}$ *is moreover infinite, then* G *has at least the same cardinality as* $\mathbf{k}$*.*

Proof. The set G is in bijection with the set S of points of degree 7 in $\mathbb{P}^2_{\mathbf{k}}$ in general position up to the action by $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$. Lemma 4.3 implies that S is non-empty. If $\mathbf{k}$ is infinite, a nodal cubic C_P contains $|\mathbf{k}|$-many points of degree 7 by Lemma 4.3. The stabiliser of a nodal cubic is finite, so the set S has the same cardinality as $|\mathbf{k}|$. □

Remark 4.5. Suppose that $\mathbf{k}$ admits a Galois extension $L/\mathbf{k}$ such that $\mathrm{Gal}(L/\mathbf{k})$ has an orbit in L of cardinality 7. Then there are non-empty relations 2.2.1. Indeed, in Lemma 4.3, we take a point $b \in \mathbf{k}$ such that $b^2 \neq \frac{1}{a_1 \cdots a_7}$.

4.2. **The set of Geiser involutions of $\mathcal{Q}$ up to automorphisms.** Let $\mathcal{Q} \subset \mathbb{P}^3_{\mathbf{k}}$ be a rational quadric surface over $\mathbf{k}$ with $\rho(\mathcal{Q}) = 1$.

Remark 4.6. The base-point of any Sarkisov link $\chi\colon \mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathcal{Q}$ of type II is a point of degree 2. Looking at relation 2.2.2, we see that the set of Geiser involutions of $\mathcal{Q}$ up to conjugation with $\mathrm{Aut}(\mathcal{Q})$ is at least as large as the set of Sarkisov links $\mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ of type II up to conjugation with $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ with a base-point of degree 6 that has no geometric component on the line through the base-point of χ^{-1}.

If $F/\mathbf{k}$ is the splitting field of a point of degree 2 in $\mathbb{P}^2_{\mathbf{k}}$, then $\mathcal{Q}_F \simeq \mathbb{P}^1_F \times \mathbb{P}^1_F$. Any two points of degree 2 in $\mathbb{P}^2_{\mathbf{k}}$ with splitting field F can be sent onto one another by an element of $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ [9, Lemma 6.11], so we may assume that it is contained in C_P.

Lemma 4.7. *Suppose that $\mathbf{k}$ has a Galois extension $L/\mathbf{k}$ such that $\mathrm{Gal}(L/\mathbf{k})$ has an orbit $a_1,\dots,a_6 \in L$ of cardinality 6. Suppose that the points $(a_i, \frac{P(a_i,1)}{a_i})$, $i = 1,\dots,6$, are not in general position.*

If $\mathbf{k}$ is finite, then $(\lambda a_i, \frac{P(\lambda a_i,1)}{\lambda a_i})$, $i = 1,\dots,6$, are in general position for all $\lambda \in \mathbf{k}^*$, except perhaps when $\lambda^6 = 1$.

If $\mathbf{k}$ is infinite, then there are $|\mathbf{k}|$-many $\lambda \in \mathbf{k}^*$ such that the $(\lambda a_i, \frac{P(\lambda a_i,1)}{\lambda a_i})$, $i = 1,\dots,6$, are in general position.

Proof. Let $p_i := (a_i, \frac{P(a_i,1)}{a_i})$ and $q_i = (\lambda a_i, \frac{P(\lambda a_i,1)}{\lambda a_i})$ for $i = 1,\dots,6$ and $\lambda \in \mathbf{k}^*$. If $p_1,\dots,p_6$ are on a conic, then $a_1 \cdots a_6 = 1$ by Lemma 4.1.

If $\mathbf{k}$ is finite, then $\mathrm{Gal}(L/\mathbf{k})$ contains a cyclic subgroup of order 6 acting non-trivially on $\{a_1,\dots,a_6\}$. In that case, no three of the p_i are collinear: indeed, if p_1,p_2,p_3 are collinear, then $a_1a_2a_3 = 1$ by Lemma 4.1. There exists $\sigma \in \mathrm{Gal}(L/\mathbf{k})$ of order 6 such that $\sigma^i(a_1) = a_i$ for $i = 1,\dots,6$, and then $a_2a_3a_4 = 1$. This means that $a_1 = a_4$, which is false. So, if $p_1,\dots,p_6$ are not in general position and $q_1,\dots,q_6$ are not in general position, then each batch is on a conic and hence $\lambda^6 = 1$.

Suppose that $\mathbf{k}$ is infinite. If p_{i_1},p_{i_2},p_{i_3} are collinear, then $a_{i_1}a_{i_2}a_{i_3} = 1$ by Lemma 4.1. If q_{j_1},q_{j_2},q_{j_3} are collinear, then λ satisfies the equation $(t^6a_1\cdots a_6 - 1)\prod_{i_1,i_2,i_3}(t^3a_{i_1}a_{i_2}a_{i_3} - 1) = 0$. Since $\mathbf{k}$ is infinite, it follows that there are $|\mathbf{k}|$-many $\lambda \in \mathbf{k}^*$ such that the q_i are in general position. □

Lemma 4.8. *Suppose that $\mathbf{k}$ has a Galois extension $L/\mathbf{k}$ such that $\mathrm{Gal}(L/\mathbf{k})$ has an orbit $a_1,\dots,a_6 \in L$. Suppose that the points $p_i = (a_i, \frac{P(a_i,1)}{a_i})$ are in general position. Then no p_i is on the line through a point in C_P of degree 2.*

Proof. The geometric components of a point of degree 2 in C_P are of the form $q_1 := (b_1, \frac{P(b_1,1)}{b_1})$, $q_2 := (b_2, \frac{P(b_2,1)}{b_2})$, where $b_1,b_2 \in \overline{\mathbf{k}}$ make up a $\mathrm{Gal}(F/\mathbf{k})$-orbit, where $F/\mathbf{k}$ is a quadratic extension. If some p_i is on the line through q_1,q_2, then $a_ib_1b_2 = 1$ by Lemma 4.1 and hence $a_i = \frac{1}{b_1b_2} \in \mathbf{k}$, impossible. □

Proposition 4.9. *Suppose that $\mathbf{k}$ has a Galois extension $L/\mathbf{k}$ such that $\mathrm{Gal}(L/\mathbf{k})$ has an orbit of cardinality 6 and let G be the set of Geiser involutions of $\mathcal{Q}$ with a base-point of degree 6 up to conjugation with $\mathrm{Aut}(\mathcal{Q})$. If $|\mathbf{k}| \neq 7$, then G is non-empty. If $\mathbf{k}$ is moreover infinite, then G is as large as $\mathbf{k}$.*

Proof. If $|\mathbf{k}| \neq 7$, then not every element of $\mathbf{k}^*$ is a 6th root of unity. Then, by Lemma 4.7, there is a Galois-orbit $p_1,\dots,p_6$ in $C_P(L)$ that is in general position in $\mathbb{P}^2_L$. By Remark 4.6, there exists a field extension $F/\mathbf{k}$ of degree 2 and C_P contains a point q of degree 2. By Lemma 4.8, none of the p_i is on the line through the point q. The Sarkisov link $\chi\colon \mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathcal{Q}$ with base-point q sends p onto a point in general position in $\mathcal{Q}$. In particular, G is non-empty.

Suppose that $\mathbf{k}$ is infinite. By Lemma 4.7 we find $|\mathbf{k}|$-many points of degree 6 on a nodal cubic that are in general position. Only finitely many elements of $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ stabilise a nodal cubic, so there are $|\mathbf{k}|$-many points of degree 6 in $\mathbb{P}^2_{\mathbf{k}}$ in general position up to $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$. Remark 4.6 and Lemma 4.8 imply that G is as large as $\mathbf{k}$. □

4.3. **The set of Geiser involutions of a del Pezzo surface of degree 6 up to automorphisms.** In this section, we study Geiser involutions on a del Pezzo surface $\mathcal{S}_6$ of degree 6 over a perfect field such that $\rho(\mathcal{S}_6) = 1$.

Remark 4.10. The blow-up of a rational point in $\mathcal{S}_6$ induces a Sarkisov link $\chi\colon \mathcal{S}_6 \dashrightarrow \mathcal{Q}$ of type II, whose inverse has a base-point p of degree 3, where $\mathcal{Q} \subset \mathbb{P}^3_{\mathbf{k}}$ is a quadric surface with $\rho(\mathcal{Q}) = 1$. Looking at relation 2.2.4, we see that the set of Geiser involutions of $\mathcal{S}_6$ up to conjugation with $\mathrm{Aut}(\mathcal{S}_6)$ is at least as large as the set Sarkisov links $\mathcal{Q} \dashrightarrow \mathcal{Q}$ of type II up to conjugation by $\mathrm{Aut}(\mathcal{Q})$ with a base-point of degree 4 in general position with a point of degree 3 in $\mathcal{Q}$ isomorphic to p.

There is a Sarkisov link $\chi'\colon \mathcal{Q} \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ such that the point $\chi'(p)$ is of degree 3, and $\chi'(p)$ can be sent by an automorphism onto a point of degree 3 in C_P [9, Lemma 6.11].

Remark 4.11. Suppose that there is Galois extensions $L/\mathbf{k}$ with Galois orbit a_1, a_2, a_3. If the points $(a_i, \frac{P(a_i,1)}{a_i})$ are collinear and the points $(\lambda a_i, \frac{P(\lambda a_i,1)}{\lambda a_i})$ are collinear for some $\lambda \in \mathbf{k}^*$, then $a_1a_2a_3 = 1 = \lambda^3 a_1a_2a_3$ by Lemma 4.1 and so $\lambda^3 = 1$.

Since $\mathcal{Q} \subset \mathbb{P}^3_{\mathbf{k}}$ is a quadric surface with $\rho(\mathcal{Q}) = 1$, there exists a quadratic extension $F/\mathbf{k}$ such that $\mathcal{Q}_F \simeq \mathbb{P}^1_F \times \mathbb{P}^1_F$.

Lemma 4.12. *Suppose that there are Galois extensions $L/\mathbf{k}$ and $N/\mathbf{k}$ with Galois orbits a_1, a_2, a_3 and b_1, b_2, b_3, b_4, respectively, and a quadratic extension $F/\mathbf{k}$ with an orbit c_1, c_2. We write $p_i = (a_i, \frac{P(a_i,1)}{a_i})$ and $q_j = (b_j, \frac{P(b_j,1)}{b_j})$ and $r_i = (c_i, \frac{P(c_i,1)}{c_i})$, and suppose that the p_i are not on a line. Then the p_i and q_i are in general position in $\mathbb{P}^2_{LN}$, and none of them are on the line passing through r_1, r_2.*

Proof. Let D be a line. If $p_i, r_1, r_2 \in D$, then all p_j are on D, contradicting our hypothesis. If $q_i, r_1, r_2 \in D$, then all the q_j are on D, which contradicts $D \cdot C_P = 3$.

If three of the q_i are on D, then $\mathrm{Gal}(N/\mathbf{k})$ preserves D and hence all q_i are on D, which is against $D \cdot C_P = 3$.

The group $\mathrm{Gal}(L/\mathbf{k})$ contains a cyclic subgroup H of order 3 by Remark 4.2. If H fixes $b_1, \ldots, b_4$, we conclude that no three of $p_1, p_2, p_3, q_1, \ldots, q_4$ are collinear because p_1, p_2, p_3 are not collinear.

Suppose now that H fixes b_1 and acts on $\{b_2, b_3, b_4\}$ non-trivially. Pick a non-trivial element $\sigma \in H$. Up to re-ordering p_1, p_2, p_3 and q_2, q_3, q_4, we can assume that $\sigma(p_1) = p_2$ and $\sigma(q_2) = q_3$.

If $p_1, p_2, q_1 \in D$, then $D = \sigma(D)$, so H preserves D, which is against the hypothesis that p_1, p_2, p_3 are not collinear.

If $p_1, p_2, q_2 \in D$, then $p_2, p_3, q_3 \in \sigma(D)$ and $p_1, p_3, q_4 \in \sigma^2(D)$. By Lemma 4.1 we have $1 = a_1a_2b_2 = a_2a_3b_3 = a_1a_3b_4$ and hence $1 = a_1^2a_2^2a_3^2b_2b_3b_4$. Then $b_2b_3b_4 \in \mathbf{k}$ and hence $b_1 \in \mathbf{k}$, which is false.

If $p_1, q_1, q_2 \in D$, then $p_2, q_1, q_3 \in \sigma(D)$ and $p_3, q_1, q_4 \in \sigma^2(D)$. By Lemma 4.1 we have $1 = a_1b_1b_2 = a_2b_1b_3 = a_3b_1b_4$ and hence $1 = b_1^3a_1a_2a_3b_1b_2b_3$. This implies that $b_1^2 \in \mathbf{k}$, which is false.

If $p_1, q_2, q_3 \in D$, then $p_2, q_3, q_4 \in \sigma(D)$ and $p_3, q_2, q_4 \in \sigma^2(D)$. By Lemma 4.1 we have $1 = a_1b_1b_2 = a_2b_3b_4 = a_3b_2b_4$ and hence $1 = a_1a_2a_3b_2^2b_3^2b_4^2$. Then $b_2^2b_3^2b_4^2 \in \mathbf{k}$ and hence $b_1^2 \in \mathbf{k}$.

If $p_1, p_2, p_3, q_1, q_2, q_3$ are on a conic D, then $a_1a_2a_3b_1b_2b_3 = 1$ by Lemma 4.1 and hence $b_1b_2b_3 \in \mathbf{k}$ and so $b_4 \in \mathbf{k}$, which is false. If $p_1, p_2, q_1, \ldots, q_4$ are on a conic D, then $a_1a_2 \in \mathbf{k}$ and hence $a_3 \in \mathbf{k}$, again false. □

Proposition 4.13. *Suppose that $\mathbf{k}$ has a Galois extension $L/\mathbf{k}$ with a Galois orbit of cardinality 4. Let G be the set of Geiser involutions of $\mathcal{S}_6$ up to conjugation by $\operatorname{Aut}(\mathcal{S}_6)$. If $|\mathbf{k}| \neq 4$, then G is non-empty. If $\mathbf{k}$ is moreover infinite, then G is at least as large as $\mathbf{k}$.*

Proof. By hypothesis, $\mathcal{S}_6$ is rational, so there exists a Sarkisov link $\mathcal{Q} \dashrightarrow \mathcal{S}_6$ whose base-point is of degree 3, so there exists a Galois extension $N/\mathbf{k}$ such that $\operatorname{Gal}(N/\mathbf{k})$ has an orbit of cardinality 3. By Remark 4.11, we find a point p of degree 3 in $C_P \subset \mathbb{P}^2_{\mathbf{k}}$ in general position because not all elements of $\mathbf{k}$ are third roots of unity. By Lemma 4.12, there is a point q of degree 4 on C_P that is in general position with p. There is a quadratic extension $F/\mathbf{k}$, so we can find a point r on C_P of degree 2. Consider the Sarkisov link $\chi'\colon \mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathcal{Q}$ with base-point r. By Lemma 4.12, the points p, q are not on the line through r, and so $\chi'(p), \chi'(q)$ are in general position in $\mathcal{Q}$. By Remark 4.10, G is non-empty.

Suppose that $\mathbf{k}$ is infinite. There are $|\mathbf{k}|$-many points of degree 4 and 3 in C_P in general position by Remark 4.11 and Lemma 4.12, and they are all not on the line through a fixed point r in C_P of degree 2 by Lemma 4.12. A nodal cubic is preserved by only finitely many automorphisms of $\mathbb{P}^2_{\mathbf{k}}$, so G is as large as $\mathbf{k}$. □

4.4. **Geiser involutions of a del Pezzo surface of degree 5.** Let $\mathcal{S}_5$ be a rational del Pezzo surface of degree 5 over a perfect field $\mathbf{k}$ with $\rho(\mathcal{S}_5) = 1$. Since $\mathcal{S}_5$ is rational, any rational point in $\mathcal{S}_5$ gives rise to a Sarkisov link $\mathcal{S}_5 \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ of type II, whose inverse has a base-point p of degree 5.

Remark 4.14. Looking at relation 2.2.3, we see that the set of Geiser involutions of $\mathcal{S}_5$ up to conjugation of $\operatorname{Aut}(\mathcal{S}_5)$ is at least as large as the set T of Sarkisov links $\chi\colon \mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ of type II up to conjugation with $\operatorname{Aut}(\mathbb{P}^2_{\mathbf{k}})$ with a base-point of degree 3 in general position with a point of degree 5 isomorphic to p.

Lemma 4.15. *Let p be a point in $\mathbb{P}^2$ of degree 5 in general position, let $L/\mathbf{k}$ be its splitting field and suppose that p is contained in a nodal cubic C_P. Suppose that there exists a Galois extension $N/\mathbf{k}$ with $\operatorname{Gal}(N/\mathbf{k})$-orbit b_1, b_2, b_3. If the p and the $(b_j, \frac{P(b_j,1)}{b_j})$ are not in general position, then for all $\lambda \in \mathbf{k}^*$ the points p and the $(\lambda b_j, \frac{P(\lambda b_j,1)}{\lambda b_j})$ are in general positon except perhaps when $\lambda^3 = 1$.*

Proof. Let $p_i = (a_i, \frac{P(a_i,1)}{a_i})$ for $i = 1, \ldots, 5$ be the geometric components of p. Let $q_i = (b_i, \frac{P(b_i,1)}{b_i})$ for $i = 1, 2, 3$. The points q_1, q_2, q_3 are aligned if and only if $b_1b_2b_3 = 1$ by Lemma 4.1. If moreover the $(\lambda b_j, \frac{P(\lambda b_j,1)}{\lambda b_j})$ are collinear, then $\lambda^3 = 1$.

By Remark 4.2, the group $\operatorname{Gal}(L/\mathbf{k})$ contains a cyclic subgroup H of order 5 and H acts trivially on $\{q_1, q_2, q_3\}$. Let D be a line in $\mathbb{P}^2_{\mathbf{k}}$ containing p_1, q_1, q_2. Then for any $\sigma \in H$, the line $\sigma(D)$ contains q_1, q_2. So $\sigma(D) = D$ for any $\sigma \in H$, but then D contains all five p_i, which is against $D \cdot C_P = 3$. Let D be a line in $\mathbb{P}^2_{\mathbf{k}}$ containing q_1, p_1, p_2. There exists $\sigma \in H$ such that $\sigma(p_1) = p_2$. The line $\sigma(D)$ contains q_1, p_2, so $\sigma(D) = D$. Since σ generates H, it follows that D contains all five p_i, which is impossible.

Let D be a conic in $\mathbb{P}^2_{\mathbf{k}}$ containing $p_1, \ldots, p_5, q_1$. Then D is defined over $\mathbf{k}$ and hence is invariant under $\mathrm{Gal}(L/\mathbf{k})$. Thus it contains all three q_i, which is against $D \cdot C_P = 6$. Let D be a conic in $\mathbb{P}^2_{\mathbf{k}}$ containing $p_1, \ldots, p_4, q_1, q_2$. For any $\sigma \in H$, the conic $\sigma(D)$ contains three of the $p_1, \ldots, p_4$, so $\sigma(D) = D$. Since σ generates H, it follows that D contains all five p_i, again impossible. Let D be a conic containing $p_1, p_2, p_3, q_1, q_2, q_3$. There exists $\sigma \in H$ such that $\sigma(p_1) = p_2$ and $\sigma(p_2) = p_3$, and then $\sigma(D) = D$. Since σ generates H, it follows that D contains all p_i, again impossible. □

Proposition 4.16. *Suppose that* $\mathbf{k}$ *has a Galois extension* $L/\mathbf{k}$ *with Galois orbit of cardinality* 3 *and suppose that the base-point of a link* $\mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathcal{S}_5$ *is on a nodal cubic isomorphic to some* C_P. *Let* G *be the set of Geiser involutions of* $\mathcal{S}_5$ *up to conjugation with* $\mathrm{Aut}(\mathcal{S}_5)$. *If* $|\mathbf{k}| \neq 4$, *then* G *is non-empty. If* $\mathbf{k}$ *is moreover infinite, then* $|G|$ *is as large as* $\mathbf{k}$.

Proof. By Remark 4.14, the set G is at least as large as the set of Sarkisov links $\chi \colon \mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$ of type II up to conjugation with $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ with a base-point of degree 3 in general position with p. By Lemma 4.15 there is a point q of degree 3 in $\mathbb{P}^2_{\mathbf{k}}$ in general position with p as long as not every element of $\mathbf{k}^*$ is a third root of unity. This is the case if $|\mathbf{k}| \neq 4$ and it follows from Remark 4.14 that G is non-empty.

If $\mathbf{k}$ is infinite, then there are $|\mathbf{k}|$-many points of degree 3 on C_P that are in general position with p by Lemma 4.15. There are only a finite number of automorphisms of $\mathbb{P}^2_{\mathbf{k}}$ that stabilise C_P, so G is at least as large as $\mathbf{k}$. □

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