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Homomorphisms from and of Cremona groups

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Contents

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3. C. Urech, S. Zimmermann: Continuous automorphisms of Cremona groups. IJM, vol. 32, 2021.
4. J. Blanc, S. Lamy, S. Zimmermann: Quotients of higher dimensional Cremona groups. Acta Math. Vol 226, no.2 (2021), 211–318.
5. S. Lamy, S. Zimmermann: Signature morphisms from the Cremona group over a non-closed field. J. Eur. Math. Soc. 22 (2020), 3133–3173.
6. C. Urech, S. Zimmermann: A new presentation of the plane Cremona group. Proc. of the AMS, vol 147, no. 7, 2019, 2741–2755.
7. T. Ducat, I. Hedén, S. Zimmermann: The decomposition groups of plane conics and plane rational cubics. Math. Res. Lett., 26(1) (2019), 35–52.
8. M.F. Robayo, S. Zimmermann: Infinite algebraic subgroups of the real Cremona group. Osaka J. of Math., vol.55, no.4 (2018), 681–712.
9. J. Blanc, S. Zimmermann: Topological simplicity of the Cremona groups. Amer. J. Math. 140 (2018), no. 5, 1297–1309.
10. S. Zimmermann: The Abelianisation of the real Cremona group. Duke Math. J. vol. 167, no.2 (2018), 211–267.
11. I. Hedén, S. Zimmermann: The Decomposition group of a line. Proc. Amer. Math. Soc. 145 (2017), no. 9, 3665–3680.
12. S. Zimmermann: The Cremona group is compactly presented. J. of the London Math. Soc., **93**, no.1 (2016), 25–46.

Preprints:

1. H.-Y. Lin, E.Shinder, S. Zimmermann: Factorisation centers in dimension two and the Groethendieck ring of varieties. arXiv:2012.04806, submitted
2. H. Kraft, A. Regeta, S. Zimmermann: Small G-varieties. arXiv:2009.05559, submitted

3. S. Asgarli, K.-W. Lai, M. Nakahara, S. Zimmermann: Biregular Cremona transformations of the plane. arXiv:1910.05302, submitted

Prices

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I Introduction

The Cremona group is the group $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ of birational self-maps of the projective space $\mathbb{P}^n_{\mathbf{k}}$ over some field $\mathbf{k}$. It is named after LUIGI CREMONA and his works [Cre63, Cre65]. In projective coordinates, the elements of $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ are of the form

$$[x_0 : \cdots : x_n] \dashrightarrow [f_0(x_0, \ldots, x_n) : \cdots : f_n(x_0, \ldots, x_n)],$$

where $f_0, \ldots, f_n \in \mathbf{k}[x_0, \ldots, x_n]$ are non-constant homogeneous polynomials of equal degree. If the f_i have no common divisors, the transformation contracts the set of hypersurfaces defined by $\det(\frac{\partial f_i}{\partial x_j}) = 0$, and it is not defined at the points $f_0 = \cdots = f_n = 0$. The transformation is given by the linear system generated by the f_i, and we can define its degree to be $\deg(f_i)$. In affine coordinates the transformations are given by quotients of polynomials

$$(x_1, \ldots, x_n) \dashrightarrow \left(\frac{p_1(x_1, \ldots, x_n)}{q_1(x_1, \ldots, x_n)}, \ldots, \frac{p_n(x_1, \ldots, x_n)}{q_n(x_1, \ldots, x_n)}\right)$$

where $p_1, \ldots, p_n, q_1, \ldots, q_n \in \mathbf{k}[x_1, \ldots, x_n]$, $q_1, \ldots, q_n \neq 0$. We have $\mathrm{Bir}(\mathbb{P}^1_{\mathbf{k}}) \simeq \mathrm{Aut}(\mathbb{P}^1_{\mathbf{k}})$, but if $n \geqslant 2$ the group $\mathrm{Bir}(\mathbb{P}^n)$ is very large and contains the group of polynomial automorphisms of the affine space of dimension n.

Cremona groups have been studied throughout the last 160 years. By Zariski's theorem, we know that any birational map between smooth projective surfaces defined over an algebraically closed field contracts only rational curves. This is reflected in the Noether-Castelnuovo theorem [Cas01, Ale16, Giz82] (re-proven by J.W. ALEXANDER and M. GIZATULLIN). It states that if $\mathbf{k}$ is algebraically closed, then $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is generated by $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ and the quadratic involution $[x : y : z] \mapsto [yz : xz : xy]$, which contracts three lines. Over a non-closed field and in higher dimension, birational maps may contract non-rational curves. Indeed, for each irreducible polynomial $p \in \mathbf{k}[y]$ the birational map $f_p \colon (x, y) \mapsto (xp(y), y)$ contracts a curve birational to $\Gamma \times \mathbb{P}^1$ onto $\{0\} \times \Gamma$, where $\Gamma \subset \mathbb{P}^1$ is the closed point given by $p = 0$. Similarly, for $n \geqslant 3$ and for any irreducible polynomial $p \in \mathbf{k}[x_2, \ldots, x_n]$, the birational map $(x_1, \ldots, x_n) \mapsto\dashrightarrow (x_1 p(x_2, \ldots, x_n), x_2, \ldots, x_n)$ contracts a hypersurface birational to $\Gamma \times \mathbb{P}^1$ onto $\{0\} \times \Gamma$, where $\Gamma \subset \mathbb{P}^{n-2}$ is the irreducible hypersurface given by $p = 0$. A classical result due to H. HUDSON and I. PAN [Hud27, Pan99] states that $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$, $n \geqslant 3$ is not generated by $\mathrm{Aut}(\mathbb{P}^n_{\mathbf{k}})$ and any set of elements of $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ of bounded degree. The argument is precisely on the above examples: we need at least as many generators as birational classes of hypersurfaces of $\mathbb{P}^{n-2}$. The same statement holds for $n = 2$ over non-closed fields. I.V. ISKOVKIKH produced in [Isk91] a generating set of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ over a perfect field $\mathbf{k}$ that is very large (it contains all of the above maps) but still quite reasonable. However, in higher dimension, no reasonable generating set of the Cremona group is known up to date. Birational Geometry in dimension $n \geqslant 3$ is much more involved than it is for surfaces, and many properties of plane Cremona group are only understood for special families of birational maps of $\mathbb{P}^n$.

The generating set of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ produced in [Isk91] makes use of the so-called *Sarkisov program*, established by A. CORTI in [Cor95] after an unpublished suggestion by SARKISOV in [Sar89] and M. REID in [Rei91]. (Well, to be precise, [Isk91] does not really use it, but all the ideas of the Sarkisov program are already well visible.) The Sarkisov program in dimension 2 provides a way to decompose any birational map between minimal geometrically rational surfaces over a perfect field into isomorphisms and special types of birational maps called *Sarkisov links*, which are well understood for surfaces. I.V. ISKOVKIKH classified all Sarkisov links between surfaces in [Isk96], and also described relations between them. [Cor95] generalises the Sarkisov program for birational maps between complex threefolds that are *Mori fibre spaces* (projective spaces are Mori fibre spaces). The algorithm is quite involved and it seems unfeasible to generalise it to dimension$\geqslant 4$.

Mori fibre spaces are outputs of the Minimal Model program (MMP). The *two-rays game* is a step in the MMP where only two extremal contractions are possible; if they lead to Mori fibre spaces, the induced birational map between the two Mori fibre spaces is a Sarkisov link. In [BCHM10], termination of the MMP in higher dimension is established in a general setting (for klt pairs). Based on the same ideas, C.D. HACON and J. MCKERNAN showed in [HM13], that any birational map between Mori fibre spaces can be decomposed into Sarkisov links. The proof does not suggest any algorithm, so the Sarkisov program in higher dimension is not really a program, but I will use the term anyway.

A question that goes back to Enriques is whether the Cremona groups are simple. S. CANTAT and S. LAMY showed in [CL13] that $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is not simple and A. LONJOU generalised this to $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ over an arbitrary field $\mathbf{k}$ in [Lon16]. To do this, they show that the Cremona group acts by isometries on an infinite-dimensional hyperbolic space, and use small-cancellation to produce normal subgroups. Unfortunately, it is not clear how to generalise the construction of this infinite-dimensional hyperbolic space in dimension$\geqslant 3$. Finding a non-trivial homomorphism of groups starting from $\mathrm{Bir}(\mathbb{P}^n)$ from scratch is not straightforward at all. In the collaboration [LSZ20] with H.-Y. LIN and E. SHINDER we show that the most intuitive homomorphism starting from $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$, given by factorisation centers over a non-closed perfect field $\mathbf{k}$ (see definition in §III.1.3), is trivial, and that it is, surprisingly, not evident. In the collaboration [BLZ21] with J. BLANC and S. LAMY, we use the Sarkisov program established in [HM13] to show that the Cremona groups in higher dimension are not simple as well, and we extend this result to large families of varieties with a $\mathbb{P}^1$-fibration. More precisely, for any Mori fibre space X of dimension $n \geqslant 3$, we construct a homomorphism of groups $\mathrm{Bir}(X) \longrightarrow \ast_I \oplus_J \mathbb{Z}/2$, where I, J are certain index sets, and show that it is non-trivial for $X = \mathbb{P}^1 \times \mathbb{P}^{n-1}$ and some large families of varieties X with a $\mathbb{P}^1$-fibration, making its kernel a non-trivial strict normal subgroup of $\mathrm{Bir}(X)$. Similar homomorphisms are also constructed in [BY20, LZ20, Sch19, Zim18a].

On can also consider group homomorphisms from $\mathrm{Bir}(X)$ to itself, and in particular group automorphisms of $\mathrm{Bir}(X)$. For $X = \mathbb{P}^2_{\mathbb{C}}$, they are all of the form $g \mapsto fg^\alpha f^{-1}$, where $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ and α is a field homomorphism of $\mathbb{C}$ acting on the coefficients of the coordinates of g [Dés06b]. In [UZ21] we generalise this result to automorphisms of $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$

that are homeomorphisms with respect to the Zariski topology, and where $\mathbf{k}$ is a field of characteristic zero, and obtain a similar result for $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$, if $\mathbf{k}$ is infinite and perfect (but not necessarily of characteristic zero).

The Sarkisov program can be interesting for the classification of Mori fibre spaces up to birational maps. For instance, given a Mori fibre space X with certain properties, one can attempt to show that there is no Sarkisov link starting from X preserving that property. The property can be the action of an algebraic group G, and the Sarkisov links is then asked to be G-equivariant, and such a Mori fibre space X is called G-birationally superrigid. This concept is used to study algebraic groups actions by birational transformations on a variety X up to conjugacy. If G is finite, this is very hard, already for surfaces. Over an algebraically closed field, the classification of finite subgroups of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ was achieved by I.V. Iskovskikh and I. Dolgachev, and by J.Blanc, culminating in [BB04, Bla09a, DI09b]. Over non-closed fields, only partial classifications of finite subgroups of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ exist [DI09a, Rob16, Yas16, Tsy13]. In dimension 2, (not necessarily finite) algebraic subgroups acting birationally on $\mathbb{P}^2$ have been classified up to conjugation and inclusion in [Bla09b] over an algebraically closed field. [Fon20] studies the connected automorphism groups of surfaces X in dimension 2, and shows that are not always contained in a maximal algebraic group acting birationally on X. In [SZ21], in collaboration with J. Schneider, we classify the infinite algebraic groups acting on $\mathbb{P}^2_{\mathbf{k}}$ over an arbitrary perfect field up to inclusion and conjugacy. It is a generalisation of the classification obtained in [RZ18] over $\mathbb{R}$. Mori fibre spaces are quite well understood in dimension $3, 4$, because a general fibre of a Mori fibre space X/B is a *Fano variety* of dimension $d = \dim X - \dim B$, and they are classical objects for $d = 2$ and classified in dimension $d = 3$ in the smooth case. This is one of the reasons why there are also some very nice partial classifications of finite groups acting birationally on threefolds, see [BCDP18, CS19, Pro11, Pro12, Pro15, PS20] for a small selection of results. Connected algebraic groups acting on $\mathbb{P}^3$ have been classified up to conjugacy and inclusion [Ume80, Ume82a, Ume82b, Ume85], and the classification is partially reproven in [BFT17, BFT19]. In dimension $n \geqslant 4$, an attack has been started in [BF20].

Even though the Sarkisov program has been established in higher dimension, the Sarkisov links themselves are not well understood. Given an explicit Mori fibre space X/B, it is in general hard to compute which Sarkisov links start from X. This is much easier in dimension 2 over a perfect field, where Mori fibre spaces are well understood and where we have a complete description of Sarkisov links [Cor95, Isk96], that is, there is an explicit list of Sarkisov links between geometrically rational Mori fibre spaces of dimension 2 together with the induced linear system of curves. In [BLZ21], we describe completely a certain type of Sarkisov link between Mori fibre spaces whose general fibre is a curve. For threefolds there are descriptions of certain Sarkisov links [BY20, BFT17, BFT19], but in general we do not have any general description, up to date. If the general fibre of the Mori fibre space X is a Fano variety of dimension $d \geqslant 3$, the description of any Sarkisov link starting from X may need good understanding of Fano varieties in dimension d.

This is a synthesis of my collaborations [BLZ21, LZ20, LSZ20, SZ21, UZ21, Zim18b] on properties of the Cremona group in dimension 2 over a perfect field and in dimension$\geqslant 3$ over $\mathbb{C}$.

Organisation of the habilitation

In §II.1–II.2, I will explain that in dimension $n \geqslant 2$ any relation between Sarkisov links is obtained by conjugating and composing so-called *elementary relations*. This is a result from [BLZ21] over $\mathbb{C}$ in dimension $\geqslant 3$ and from [LZ20] over a perfect field $\mathbf{k}$ in dimension 2.

In §II.2–II.4 I give the description of elementary relations among Sarkisov links obtained in [Sch19] (dimension 2 over a perfect field) and [BLZ21] (dimension$\geqslant 3$ over $\mathbb{C}$) that include a special type of Sarkisov link preserving a $\mathbb{P}^1$-fibration. I also give the description of the elementary links in [LZ20] including a Bertini link (only trivial relations) and the elementary relations preserving a fibration of del Pezzo surfaces over a curve described in [BY20].

In §III.1–§III.2, I describe for some special families of varieties X the surjective homomorphism of groups $\mathrm{Bir}(X) \longrightarrow \ast_I \bigoplus_J \mathbb{Z}/2$ constructed in [BLZ21, BY20, LZ20, Sch19]. In fact, all of them are obtained as follows: the description of the elementary relations from §II.2–II.4 give rise to a homomorphism of groupoids $\mathrm{BirMori}(X) \longrightarrow \ast_{I'} \oplus_{J'} \mathbb{Z}/2$, which we then restrict and cut off factors in the target group. In §III.1.3, we also explain the result of [LSZ20], which states that for a smooth projective surface over a perfect field, the homomorphism from $\mathrm{Bir}(X) \longrightarrow \mathbb{Z}[\mathrm{Var}^0/\mathbf{k}]$ given by factorisation centers is trivial.

In §III.3 we switch to considering homomorphism of groups from the Cremona group to itself. The Cremona groups carry a natural topology, the so-called *Zariski topology* that restricts to the usual Zariski topology on $\mathrm{Aut}(\mathbb{P}^n_{\mathbf{k}})$. The automorphisms of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ are all inner up to a field automorphism of $\mathbb{C}$ [Dés06b], and we explain the generalisation to the automorphisms of $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ that are homeomorphisms with respect to the Zariski topology if $\mathbf{k}$ is of characteristic zero obtained in [UZ21].

In §IV.1–IV.2, we present some structure results on the Cremona groups that follow from the existence of some non-trivial homomorphisms from $\mathrm{Bir}(\mathbb{P}^n)$ to free products and sums of $\mathbb{Z}/2$ explained. These are results from [BLZ21, LZ20, Zim18b].

In §V.2 we present the classification over a perfect field of del Pezzo surfaces of degree 8 and 6, and of exceptional conic bundles, and describe their automorphisms groups, as given in [SZ21].

In §V.3, we present the classification of the infinite algebraic groups acting birationally on the projective plane, up to inclusion and conjugacy, as proven in [SZ21].

The Appendix A consists of the list of elementary relations among Sarkisov links between smooth rational Mori fibre spaces. This is mostly busy-work, and the best place for such a list is a thesis, a habilitation or a book on Sarkisov links or the Cremona group. The list can also be found in [LS21].

In this manuscript we impose the following convention: in dimension $n = 2$, we work over an arbitrary perfect field unless stated otherwise, and in dimension $n \geqslant 3$ we work over $\mathbb{C}$ unless stated otherwise.

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II Sarkisov links and relations

A *Sarkisov link* is a special type of birational map between terminal *Mori fibre spaces* that were introduced by I.V. Iskovskivkh in [Isk91] and M. Reid in [Rei91] after an announcement by V.G. Sarkisov in [Sar89], which however never seemed to have been followed up by details. It is shown in [Isk96, Cor95] independently by I.V. Iskovskivkh and A. Corti that all birational maps between geometrically rational smooth projective surfaces over a perfect field decompose into Sarkisov links and automorphisms of Mori fibre spaces. In [Cor95], A. Corti shows that any birational map between Mori fibre spaces of dimension $n = 3$ over $\mathbb{C}$ decompose into Sarkisov links and automorphisms of Mori fibre spaces. The statement is generalised to any dimension by C.D. Hacon and J. McKernan in [HM13]. Let X be a variety defined over $\mathbb{C}$ and $\mathrm{BirMori}(X)$ the groupoid of birational maps between Mori fibre spaces birational to X. In this section, we give a presentation $\mathrm{BirMori}(X) = \langle \text{Sarkisov links, automorphisms} \mid R\rangle$, where R is a set of generating relations.

II.1 Rank r fibrations and Sarkisov links

We are going to introduce Sarkisov links as defined in [BLZ21] as families of *rank* 2 *fibrations* joined by a sequence of pseudo-isomorphisms. We therefore first define the notion of *rank r fibration* and then explain that rank 1 fibrations are precisely terminal Mori fibre spaces, rank 2 fibrations correspond to Sarkisov links between Mori fibre spaces and rank 3 fibrations correspond to relations between Sarkisov links, called *elementary relations*.

II.1.1 Mori dream spaces and Minimal Model Program

Before we define *Mori dream spaces*, let us recall some notions on divisors. Let X be a normal variety over $\mathbb{C}$, let $\mathrm{Div}(X)$ be the group of Cartier divisors, $\mathrm{Pic}(X) = \mathrm{Div}(X)/\sim$ the Picard group of divisors modulo linear equivalence, $N^1(X) = \mathrm{Div}(X)\otimes\mathbb{R}/\equiv$ the space of $\mathbb{R}$-divisors on X modulo numerical equivalence and $\rho(X)$ its dimension. We denote by $N_1(X)$ the dual space of 1-cycles with real coefficients modulo numerical equivalence, and we have a pairing $N^1(X) \times N_1(X) \longrightarrow \mathbb{R}$ induced by the intersection. The variety X is $\mathbb{Q}$-factorial if all Weil divisors on X are $\mathbb{Q}$-Cartier, and an element of $\mathrm{Div}(X)\otimes\mathbb{Q}$ is called $\mathbb{Q}$-divisor.

Let $m > 0$ be a sufficiently large and divisible integer. A divisor D on X is *movable* if the base-locus of the linear system $|mD|$ has codimension at least 2 and it is *big* if the rational map induced by $|mD|$ is birational onto its image. It is *semiample* if $|mD|$ is base-point free, and this is the case if D is the pullback of an ample class by a morphism. The movable cone $\overline{\mathrm{Mov}}(X)$ is the closure of the cone spanned by movable divisors, and $\mathrm{Nef}(X)$ is the cone spanned by nef divisors.

For a morphism $\pi\colon X \longrightarrow Y$ between normal varieties we denote by $NE(X/Y) \subset N_1(X/Y) \subset N_1(X)$ the cone and the subspace generated by curves contracted by π, and by $N^1(X/Y)$ the quotient of $N^1(X)$ by the orthogonal of $N_1(X/Y)$. Its dimension $\rho(X/Y)$ is the *relative Picard rank* of π. We denote by $\mathrm{Nef}(X/B)$ and $\overline{\mathrm{Mov}}(X/B)$ the images of $\mathrm{Nef}(X)$ and $\overline{\mathrm{Mov}}(X)$ in the quotient $N^1(X/Y)$. We denote by $\dashrightarrow$ a rational map and by $\cdots\!\!\rightarrow$ a *pseudo-isomorphism*, that is, a birational map that is an isomorphism outside codimension 2 sets on the domain and target variety.

A normal variety X has *rational singularities* if for some desingularisation $\pi\colon Z \longrightarrow X$ we have $R^i\pi_*\mathcal{O}_Z = 0$ for all $i > 0$, where $R^i\pi_*\mathcal{O}_Z$, $i \geqslant 0$, is the sheaf defined on each open affine subset $U \subset X$ as $R^i\pi_*\mathcal{O}_Z(U) = H^i(\pi^{-1}(U), \mathcal{O}_Z)$.

Definition II.1.1. A a surjective morphism $\eta\colon X \longrightarrow B$ between normal varieties is a *Mori dream space* if the following conditions hold:

1. X is $\mathbb{Q}$-factorial, and both X and B have rational singularities;
2. a general fibre of η is rationally connected and has rational singularities;
3. $\mathrm{Nef}(X/B)$ is the convex cone generated by finitely many semiample divisors;
4. there exist finitely many pseudo-isomorphisms $f_i\colon X \cdots\!\!\rightarrow X_i$ over B such that each X_i is a $\mathbb{Q}$-factorial variety satisfying (3) and $\overline{\mathrm{Mov}}(X/B) = \cup f_i^*(\mathrm{Nef}(X_i/B))$.

If B is a point, we get back the classical definition of a Mori dream space [HK00].

Let X/B be a surjective morphism between normal varieties whose general fibres are rationally connected. Assume that X is $\mathbb{Q}$-factorial and that X, B and the general fibres have rational singularities. Then X/B is a Mori dream space if and only if its Cox sheaf is finitely generated [BLZ21, Lemma 2.6], which is proven analogously to [KKL14, Corollaries 4.4 and 5.7].

Let me recall the notion of *D-Minimal Model program* (or D-MMP): Suppose that X is normal and $\mathbb{Q}$-factorial. Let $\pi\colon X \longrightarrow Y$ be a surjective birational morphism with connected fibres and $\rho(X/Y) = 1$. If the exceptional locus is of codimension 1, it is a prime divisor and π is called a *divisorial contraction*. It is called *small contraction* if the exceptional locus is of co-dimension$\geqslant 2$, and in that case Y is not $\mathbb{Q}$-factorial. Let $\overline{NE}(X)$ be the closure of $NE(X)$ and $C \in \overline{NE}(X)$ an *extremal class*, that is, if $C = C_1 + C_2$ with $C_1, C_2 \in \overline{NE}(X)$, then C, C_1, C_2 are proportional. The contraction of C exists if there is a surjective morphism $\pi\colon X \longrightarrow Y$ with connected fibres to a normal variety Y and $\rho(X/Y) = 1$ such that any curve contracted by π is numerically proportional to C. If π is a small contraction, we say that a *log-flip* of C exists if there is a *pseudo-isomorphism* $X \cdots\!\!\rightarrow X'$ (it is birational and does not contract any divisor nor does its inverse) over Y which is not an isomorphism such that X' is normal $\mathbb{Q}$-factorial and $X' \longrightarrow Y$ is a small contraction that contracts curves proportional to a class C'. For each $D \in N^1(X)$ if D' is the image of D unter the pseudo-isomorphism, we have a sign change between $D \cdot C$ and $D' \cdot C'$. We call $X \cdots\!\!\rightarrow X'$ a *D-flip* if $D \cdot C < 0$.

For $D \in \mathrm{Div}(X)$, a *step in the D-MMP* is the removal of an extremal class $C \in \overline{NE(X)}$ with $D \cdot C < 0$ via a divisorial contraction or via a D-flip. If D is nef on X, then X is called *D-minimal model*. If there exists a contraction $X \longrightarrow Y$ with $\rho(X/Y) = 1$, $\dim Y < \dim X$ and $-D$ relatively ample, we say that X/Y is a *D-Mori fibre space*. If D is the canonical divisor of X, it is simply called *Mori fibre space*.

Proposition II.1.2 ([HK00, Proposition 1.11], or [KKL14, Theorem 5.4])**.** *If X/B is a Mori dream space, then for any class $D \in N^1(X)$ one can run a D-MMP from X over B, and there are only finitely many possible outputs for such MMP.*

II.1.2 Rank r fibrations and Sarkisov links

Let X be a $\mathbb{Q}$-factorial normal variety, $\pi\colon Z \longrightarrow X$ a desingularitation with exceptional divisors $E_1, \ldots, E_r$. Then X is *terminal* if in the ramification formula $K_Z = \pi^* K_X + \sum a_i E_i$ we have $a_i > 0$ for all i. It is *klt* (*Kawamata log terminal*) if $a_i > -1$ for all i. This does not depend on the resolution, and being terminal is preserved under the step of the (classical) K-MMP.

Definition II.1.3 ([BLZ21, Definition 3.1])**.** Let $r \geqslant 1$ be an integer. A morphism $\eta\colon X \longrightarrow B$ is a *rank r fibration* if the following conditions hold:

1. X/B is a Mori dream space;
2. $\dim X > \dim B \geqslant 0$ and $\rho(X/B) = r$;
3. X is $\mathbb{Q}$-factorial and terminal, and for any divisor D on X the output of any D-MMP from X over B is still $\mathbb{Q}$-factorial and terminal;
4. there exists an effective $\mathbb{Q}$-divisor Δ_B such that (B, Δ_B) is klt;
5. the anticanonical divisor $-K_X$ is η-big.

The notion of rank r fibration seems to be new in literature, although it resembles the notion of other fibrations, for instance a fibration of Fano type [Bir19], but it has strong restrictions for allowed singularities. These are imposed for technical reasons; the definition is made so that rank r fibrations fulfill their purpose.

Example II.1.4 ([BLZ21, Lemma 3.3])**.** If $\eta\colon X \longrightarrow B$ is a surjective morphism between normal varieties, then X/B is a rank 1 fibration if and only if X/B is a terminal Mori fibre space.

In dimension 2, terminal means smooth and for any rank r fibration X/B, the anticanonical divisor $-K_X$ is relatively ample. Indeed, if we start the $(-K_X)$-MMP over B on a rank r fibration X/B, we cannot contract any curve, because its image would be a singular point, which contradicts Definition II.1.3(3). So, either $-K_X$ is relatively nef or there is a fibration such that $-K_X$ is negative against every fibre. Since $-K_X$ is relatively big over B by hypothesis, the latter is impossible. Now, since X is a smooth surface and $-K_X$ being relatively big and nef over B, it follows that $-K_X$ is relatively ample over B.

A surjective morphism $\eta\colon X \longrightarrow B$ from a smooth projective surface to a point or a smooth curve B with connected fibres such that $-K_X$ is η-ample and $\rho(X/B) = r$ is a rank r fibration, because (smooth) del Pezzo surfaces and (smooth) conic fibrations whose singular fibres have at most two connected components are Mori dream spaces.

Definition II.1.5 ([LZ20, §2.1])**.** Let $\mathbf{k}$ be a perfect field. A *rank r fibration* in dimension 2 over $\mathbf{k}$ is defined to be a surjective morphism $X \longrightarrow B$ from a smooth projective surface over $\mathbf{k}$ with $X(\mathbf{k}) \neq \varnothing$ to a smooth curve or point B with connected fibres such that $-K_X$ is relatively ample and $\rho(X/B) = r$.

Remark II.1.6. The argument from above can be generalised. Given a rank r fibration $\eta\colon X \longrightarrow B$, we can run a $(-K)$-MMP from X over B. The restriction on the singularities of X imply that the only possible step is log-flips and after finitely many steps we'll arrive on a rank r fibration such that $-K$ is relatively big and nef [BLZ21, Lemma 3.5]. Then for a general point $p \in B$ the fibre $\eta^{-1}(p)$ is pseudo-isomorphic to a terminal weak Fano variety [Kol97, 7.7] and the set of curves in $\eta^{-1}(p)$ that are trivial against the canonical divisor cover a subset of codimension $\geqslant 2$ in $\eta^{-1}(p)$ [BLZ21, Corollary 3.6].

We say that a rank r fibration X/B *factorises through* a rank r' fibration X'/B', or that X'/B' is *dominated by* X/B, if the fibration X/B and X'/B' fit into a commutative diagram

$$\begin{tikzcd} X \arrow[rrr] \arrow[dr, dashed] & & & B \\ & X' \arrow[r] & B' \arrow[ur] & \end{tikzcd}$$

where $X \dashrightarrow X'$ is a birational contraction and $B' \longrightarrow B$ is a morphism with connected fibres. It implies $r \geqslant r'$. If the birational map $X \dashrightarrow X'$ is a morphism, then X/B' is a rank $\rho(X/B')$ fibration. If X/B is rank r fibration and Y is obtained by performing a log-flip (resp. divisorial contraction) over B, then Y/B is a rank r fibration (resp. rank $r-1$ fibration) [BLZ21, Lemma 3.4].

Example II.1.7 ([BLZ21, Lemma 3.7], [LZ20, §2.3])**.** Let Y/B be a rank 2 fibration over $\mathbb{C}$ and $\dim Y \geqslant 2$, or over a perfect field $\mathbf{k}$ and $\dim Y = 2$. Running the two-rays game from Y over B means that there are exactly two rank 1 fibrations (= terminal Mori fibre spaces) X_1/B_1, X_2/B_2 dominated by Y/B, that both fit into a commutative diagram

$$\begin{tikzcd} {} & Y \arrow[l, dotted] \arrow[r, dotted] & {} \arrow[d] \\ {} \arrow[u, phantom] \arrow[d] & & {} \arrow[dl] \\ {} \arrow[r] & B & \end{tikzcd}$$

where the dotted arrows are sequences of log-flips, and the other four arrows are morphisms of relative Picard rank 1. The induced birational map $\chi\colon X_1 \dashrightarrow X_2$ is called *Sarkisov link*, and the diagram is called *Sarkisov diagram.* While a rank 2 fibration uniquely determines the Sarkisov diagram, it only defines the link up to taking inverse. We thus have the four types of links listed in Figure II.1, where an arrow marked with *div* is a divisorial contraction, an arrow marked *fib* is a rank 1 fibration, and the dotted arrows are sequences of log-flips.

In [Isk96, Cor95, HM13], the definition of Sarkisov links is also made to correspond to a two rays-game, but they ask for less conditions on the singularities of the fibres. In dimension $n = 2$, a link of type II over a curve B is classically called *elementary transformation.*

Example II.1.8. Let us illustrate a well-known composition of Sarkisov links over an arbitrary perfect field $\mathbf{k}$. Consider the involution $\sigma\colon [x : y : z] \mapstochar\dashrightarrow [yz : xz : xy]$ of $\mathbb{P}^2_{\mathbf{k}}$. It has three rational base-points, namely $p_1 := [1 : 0 : 0]$, $p_2 := [0 : 1 : 0]$, $p_3 := [0 : 0 : 1]$, and contracts the three lines passing through any two of the three. We can write $\sigma = \chi_4 \circ \cdots \circ \chi_1$, with $\chi_1, \ldots, \chi_4$ the Sarkisov links in the commutative diagram below,

$$\begin{tikzcd}[column sep=small] \cdots \arrow[d, "div"'] & & X_2 \arrow[d, "fib"] \\ X_1 \arrow[urr, dashed, "\chi"'] \arrow[dr, "fib"'] & & B_2 \arrow[dl] \\ & B_1 = B & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \cdots \arrow[d, "div"'] & & \cdots \arrow[d, "div"] \\ X_1 \arrow[rr, dashed, "\chi"'] \arrow[dr, "fib"'] & & X_2 \arrow[dl, "fib"] \\ & B_1 = B = B_2 & \end{tikzcd}$$

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$$\begin{tikzcd}[column sep=small] X_1 \arrow[d, "fib"'] \arrow[drr, dashed, "\chi"'] & & \cdots \arrow[d, "div"] \\ B_1 \arrow[dr] & & X_2 \arrow[dl, "fib"] \\ & B = B_2 & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] X_1 \arrow[d, "fib"'] \arrow[rr, dotted, "\chi"'] & & X_2 \arrow[d, "fib"] \\ B_1 \arrow[dr] & & B_2 \arrow[dl] \\ & B & \end{tikzcd}$$

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Figure II.1: Rank 2 fibrations correspond to any of the four types of Sarkisov links.

where the blow-ups are marked with the point that is blown up or the fibre f_p containing the point p that is contracted, or the exceptional curve $E \subset \mathbb{F}_1$ that is contracted.

$$\begin{tikzcd}[column sep=small] & & Y_1 \arrow[dl, "p_2"'] \arrow[dr, "f_{p_2}"] & & Y_2 \arrow[dl, "p_3"'] \arrow[dr, "f_{p_3}"] & & \\ & \mathbb{F}_1 \arrow[dl, "p_1"] \arrow[rr, dashed, "\chi_2"] \arrow[drr] & & \mathbb{F}_0 \arrow[d] \arrow[rr, dashed, "\chi_3"] & & \mathbb{F}_1 \arrow[dll] \arrow[dr, "\chi_4", "E"'] & \\ \mathbb{P}^2 \arrow[ur, dashed, bend left, "\chi_1"] \arrow[dr] & & & \mathbb{P}^1 \arrow[dll] \arrow[drr] & & & \mathbb{P}^2 \arrow[dl] \\ & * & & & & * & \end{tikzcd}$$

Example II.1.9 ([BLZ21, Lemma 4.2, Proposition 4.3],[LZ20, Proposition 2.6])**.** Let T/B be a rank 3 fibration that factorises through a rank 1 fibration X_1/B_1, again over $\mathbb{C}$ and $\dim T \geqslant 2$, or over a perfect field and $\dim T = 2$. Then there exist exactly two rank 2 fibrations that factorise through X_1/B_1 and that are dominated by T/B, up to pseudo-isomorphisms $X_2 \dashrightarrow X_2'$ between rank 2 fibrations X_2/B_2 and X_2'/B_2' such that there is a commutative diagram

$$\begin{tikzcd} & T \arrow[dl, dashed] \arrow[dr, dashed] & \\ X_2 \arrow[rr, dotted] \arrow[d] & & X_2' \arrow[d] \\ B_2 \arrow[rr, "\sim"] & & B_2' \end{tikzcd}$$

Since T/B is a Mori dream space, one can run any D-MMP from T and there are only finitely many outcomes of such MMP, see Proposition II.1.2. So, T/B dominates only finitely many rank 1 fibrations, and hence it dominates only finitely many Sarkisov links χ_i, and they fit into a relation

$$\chi_t \circ \cdots \circ \chi_1 = \mathrm{id}.$$

We call it an *elementary relation* between Sarkisov links dominated by T/B. It is uniquely defined by T/B up to taking inverse, cyclic permutation and insertion of isomorphisms.

We say that a group G is *generated* by a set $S \subset G$ if there is a surjective morphism of groups $\pi\colon F_S \longrightarrow G$, where F_S is the free group generated by S. A subset $R \subset \ker\pi$ is called a set of *generating relations of* G if it generates $\ker\pi$ as normal subgroups, that is, if any element of $\ker\pi$ is a composition of elements of the form $\Pi_{i=1}^k w_i r_i w_i^{-1}$, with $r_1, \dots, r_k \in R \cup R^{-1}$ and $w_1, \dots, w_k \in F_S$. The notions of morphisms, generating set and generating relations can be transferred to groupoids.

For a variety X, we denote by $\mathrm{BirMori}(X)$ the groupoid of birational maps $X' \dashrightarrow X''$ between rank 1 fibrations X'/B' and X''/B'' with X', X'' birational to X.

Theorem II.1.10 ([HM13],[Cor95],[Isk96])**.**

1. *For any projective surface X over a perfect field $\mathbf{k}$ with $X(\mathbf{k}) \neq \varnothing$, the groupoid $\mathrm{BirMori}(X)$ is generated by Sarkisov links and isomorphisms between Mori fibre spaces.*
2. *For any variety X over $\mathbb{C}$ of dimension $\geqslant 3$, the groupoid $\mathrm{BirMori}(X)$ is generated by Sarkisov links and isomorphisms between Mori fibre spaces.*

Theorem II.1.10 was proven for projective surfaces by [Cor95] by A. Corti based on an announcement in [Sar89] by V.A. Sarkisov, and reproven by V.A. Iskovskikh in [Isk96]. Based on the same idea of factorisation, V.A. Iskovskikh had produced earlier a generating set of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ over a perfect field $\mathbf{k}$ in [Isk91]. In dimension $n = 3$ over $\mathbb{C}$ the statement is proven in [Cor95], and in any dimension $n \geqslant 2$ over $\mathbb{C}$ it is shown in [HM13]. The proofs in [Isk91, Isk96, Cor95] provide an algorithm for the decomposition, while [HM13] is not algorithmic. The definition of Sarkisov links in [BLZ21, LZ20] is more restrictive than the one in [Isk91, Cor95, HM13], but the proofs of the above statement can be repeated, as is done in [BLZ21, Theorem 4.28(1)] and [LZ20, Theorem 3.1(1)], with some technical brewing in dimension $\geqslant 3$ to accommodate all conditions imposed by the notion of rank r fibrations.

By *trivial relations* between Sarkisov links we mean relations between isomorphisms and relations of the form $(\chi')^{-1}\alpha\chi = 1$, where $\chi\colon X \dashrightarrow X'$, $\chi'\colon X' \dashrightarrow X''$ are Sarkisov link and $\alpha\colon X' \xrightarrow{\sim} X''$ is an isomorphism such that $\chi' = \alpha \circ \chi$.

Theorem II.A ([BLZ21, Theorem 4.28(2)], [LZ20, Theorem 3.1(2)])**.**

1. *For a projective surface X over a perfect field $\mathbf{k}$ with $X(\mathbf{k}) \neq \varnothing$, any relation between Sarkisov links in $\mathrm{BirMori}(X)$ is generated by the trivial relations and the elementary relations.*
2. *For a variety over $\mathbb{C}$ of dimension $\geqslant 3$, any relation between Sarkisov links in $\mathrm{BirMori}(X)$ is generated by the trivial relations and the elementary relations.*

Theorem II.A is inspired by [Kal13, Theorem 1.3], where however the definition of elementary relation in [Kal13] is different from the one in [LZ20, BLZ21]. In [Kal13, p.1687] a relation $\chi_r \circ \dots \circ \chi_1 = \mathrm{id}$ of Sarkisov links $\chi_i\colon X_i/B_i \dashrightarrow X_{i+1}/B_{i+1}$, $i = 1, \dots, r$, is called an elementary relation if there exists a variety B with morphisms $B_i \longrightarrow B$ that commutes with the links and such that $\rho(X_i/B) \leqslant 3$ for all $i = 1, \dots, r$.

[IKT93] gives a presentation of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ in terms of the generating set produced in [Isk91]. The set of generators and the set of generating relations is very large because they insist for the elements to be contained in $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$, although they implicitly see the maps as compositions of Sarkisov links, as is evident from the notation they use. The list was not used in the proof of Theorem II.A(1). We list all elementary relations between Sarkisov links in dimension 2 over an arbitrary perfect field in Appendix A. In §II.3 and §II.4 we present some special examples of elementary relations in dimension 2 and $\geqslant 3$, respectively.

II.2 Why are elementary relations enough?

In this section, we explain the rough idea of why any relation among Sarkisov links in $\mathrm{BirMori}(X)$ is generated by elementary relations. For smooth projective surfaces over a perfect field with non-empty set of rational points the same idea works if made $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-equivariant, and this exactly what is done in [LZ20, §3] to prove Theorem II.A(1).

II.2.1 The set-up: seeing the MMPs on a fan $\mathcal{C}$

We first define an *ample model* of a divisor as given in [BCHM10, Definition 3.6.5]. Let Z be a terminal $\mathbb{Q}$-factorial variety, D an $\mathbb{R}$-divisor on Z and $\varphi\colon Z \dashrightarrow Y$ a dominant rational map to a normal variety Y with a resolution

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Z \arrow[rr, dashed, "\varphi"] & & Y \end{tikzcd}$$

where W is smooth, p is a birational morphism and q is a morphism with connected fibres. We say that φ is an *ample model of* D if there exists an ample divisor H on Y such that p^*D is linearly equivalent to $q^*H + E$, where $E \geqslant 0$, and for each effective $\mathbb{R}$-divisor R linearly equivalent to p^*D we have $R \geqslant E$. If φ is a birational contraction, we say that φ is a *semiample model* of D if $H = \varphi_*D$ is semiample (hence in particular $\mathbb{R}$-Cartier) and if $p^*D = q^*H + E$ where $E \geqslant 0$ is q-exceptional.

Some references ask an ample model of a $\mathbb{Q}$-divisor D to factorise through a semiample model. If it does, the ample model is the rational map φ_D associated to the linear system $|mD|$ for some $m \gg 0$, whose image is $Z_D = \mathrm{Proj}(\oplus_m H^0(Z, mD))$, where the sum is over all integers such that mD is Cartier, see [KKL14, Remark 2.4(ii)]. The ample model exists if the ring $\bigoplus_m H^0(Z, mD)$ is finitely generated, which is the case if $D = K_Z + A$ for some ample $\mathbb{Q}$-divisor, as follows from [BCHM10, Corollary 1.1.2]. If it exists, it is unique up to composing with an isomorphism, and a birational map $\varphi\colon Z \dashrightarrow Y$ is the ample model of D if and only if it is a semiample model of D and φ_*D is ample [BCHM10, Lemma 3.6.6].

The following proposition is assembled from [HM13, Lemma 4.1] and [Kal13, Proposition 3.1(ii)], see [BLZ21, Proposition 4.23].

Proposition II.2.1. *Let $t \geqslant 2$ be an integer. For $i = 1, \ldots, t$, let $\eta_i\colon X_i \longrightarrow B_i$ be a terminal Mori fibre space and let $\theta_i\colon X_i \dashrightarrow X_{i+1}$ be a birational map, where $X_{t+1} := X_1$.*

Suppose that $\theta_t \circ \cdots \circ \theta_1 = \mathrm{id}$. Then there exists a smooth variety Z together with birational morphisms $\pi_i \colon Z \longrightarrow X_i$, $i = 1, \ldots, t$, and ample $\mathbb{Q}$-divisors $A_1, \ldots, A_m$ on Z such that the following hold:

1. *the divisors $A_1, \ldots, A_m$ generate the $\mathbb{R}$-vector space $N^1(Z)$;*
2. *for $i = 1, \ldots, t$, the birational morphism π_i and the morphism $\eta_i \circ \pi_i$ are the ample models of an element of*

$$\mathcal{C} = \{a_0 K_Z + \sum_{i=1}^{m} a_i A_i \in N^1(Z) \mid a_0, \ldots, a_m \geqslant 0\} \cap \overline{\mathrm{Eff}}(Z)$$

3. *for $i = 1, \ldots, t$ we have $\theta_i \circ \pi_i = \pi_{i+1}$ (with $\pi_{t+1} := \pi_1$). We then have a commutative diagram*

$$\begin{tikzcd}
 & & X_1 \arrow[dl, dashed, "\theta_1"'] & & \\
 & X_2 \arrow[dl, dashed, "\theta_2"'] & & X_t \arrow[ul, dashed, "\theta_t"'] & \\
X_3 \arrow[dr, dashed, "\theta_3"'] & & Z \arrow[uu, "\pi_1"] \arrow[ul, "\pi_2"] \arrow[ll, "\pi_3"] \arrow[dl, "\pi_4"] \arrow[dd, "\pi_5"] \arrow[ur, "\pi_t"'] \arrow[rr, "\pi_{t-1}"] & & X_{t-1} \arrow[ul, dashed, "\theta_{t-1}"'] \\
 & X_4 \arrow[dr, dashed, "\theta_4"'] & & \iddots & \\
 & & X_5 & &
\end{tikzcd}$$

We say that two divisors D and D' on Z are *Mori equivalent* if they have the same ample model. In the situation of Proposition II.2.1 every element of $\mathcal{C}$ has an ample model, as mentioned before the statement. The Mori equivalence classes induce a partition $\mathcal{C} = \amalg_{i \in I} \mathcal{A}_i$. We call the $\mathcal{A}_i$ *Mori chambers*, and we denote by $\varphi_i \colon Z \dashrightarrow Z_i$ the common ample model of all $D \in \mathcal{A}_i$. It follows from [HM13, Theorem 3.3] that in our setting the partition is finite, $\mathcal{C}$ is a cone over a polytope, each $\mathcal{A}_i$ is a finite union of relative interiors of cones over rational polytopes (see [BLZ21, Proposition 4.11]). While [HM13, Theorem 3.3] work in $\mathrm{Div}(Z)$, we work in $N^1(Z)$ because numerically equivalent divisors belong to the same Mori chamber in $\mathrm{Div}(Z)$, see [KKL14, Lemma 3.11] for the big case and [BLZ21, Proposition 4.14(5)] for the non-big case.

The fan on $\mathcal{C}$: We say that a Mori chamber has *maximal dimension* if it spans the $\mathbb{R}$-vector space $N^1(Z)$. This is equivalent to φ_i being birational and Z_i is $\mathbb{Q}$-factorial, and equivalent to φ_i being a birational contraction that is output of a $(K_Z + \Delta)$-MMP for some $K_Z + \Delta \in \mathcal{C}$ [HM13, Theorem 3.3(3)]. The closure of the chambers of the maximal dimension yields a fan structure on $\mathcal{C}$, which is the image in $N^1(Z)$ of the fan structure considered in [KKL14, Theorem 3.2&4.2], and which in turn generalises the fan in [ELM+06]: for each Mori chamber $\mathcal{A}_i$ of maximal dimension, the closure of $\mathcal{A}_i$ is

$$\overline{\mathcal{A}}_i = \{D \in \mathcal{C} \mid \varphi_j \text{ is the semiample model of } D\}$$

and it is the intersection of $\mathcal{C}$ with the closed convex cone generated by $\varphi_i^* \mathrm{Nef}(Z_i)$ and by the exceptional divisors of φ_i. For any i, j such that $\overline{\mathcal{A}}_j \cap \mathcal{A}_i \neq \varnothing$, there exists a morphism $\varphi_{ji} \colon Z_j \longrightarrow Z_i$ with connected fibres such that $\varphi_i = \varphi_{ji} \circ \varphi_j$. If φ_j is birational (i.e. $\mathcal{A}_j$

maximal), then

$$\overline{\mathcal{A}}_j \cap \overline{\mathcal{A}}_i = \{D \in \overline{\mathcal{A}}_j \mid {\varphi_j}_* D \cdot C = 0 \text{ for each } C \in N_1(Z_j/Z_i)\}$$

[HM13, Theorem 3.3(2)] or [KKL14, Theorem 4.2(3)], see also [BLZ21, Proposition 4.41] (and [LZ20, Proposition 3.7] in dimension 2 over non-closed fields).

Let $\partial^+\mathcal{C} \subset \mathcal{C}$ be the set of non-big divisors. It is a closed subset of the boundary of $\mathcal{C}$, and $\mathcal{A}_i \subset \partial^+\mathcal{C}$ if $\dim Z_i < \dim Z$, and $\mathcal{A}_i \subset \mathcal{C}\backslash\partial^+\mathcal{C}$ if $\dim Z_i = \dim Z$.

The fan on $\mathcal{C}$ encodes MMPs: for an ample divisor Δ in $\mathcal{C}$, the successive chambers intersected by the segment $[\Delta, K_Z] \cap \mathcal{C}$ correspond to successive steps in a K_Z-MMP from Z. It is called a *K_Z-MMP with scaling* Δ in [BCHM10, Remark 3.10.10]. By perturbing Δ a little, the segment $[\Delta, K_Z]$ becomes transversal to the polyhedral decomposition. The intermediate codimension 1 faces cut by $[\Delta, K_Z]$ in $\mathcal{C}\backslash\partial^+\mathcal{C}$ are steps in the MMP corresponding to a log-flip or a divisorial contraction. They correspond to walls given by a change of sign of the intersection of $K_Z + \Delta$ with curves in an exceptional divisor of some φ_j. The intersection of $[\Delta, K_Z]$ with $\partial^+\mathcal{C}$ is the last step of the MMP.

II.2.2 Rank r fibrations correspond to special faces in $\partial^+\mathcal{C}$

Let us look more closely at a special type of codimension $r \geqslant 1$ faces in $\mathcal{C}$.

Inner faces of $\mathcal{C}$: Let $\mathcal{F}^r$ be a face of the fan $\mathcal{C}$ of codimension r, that is the codimension in $N^1(Z)$ of the smallest vector space containing it. We say that it is an *inner face* if it meets the interior of $\mathcal{C}$ or the relative interior of $\partial^+\mathcal{C}$. [BLZ21, §4.B] summarises the following facts on inner faces of $\mathcal{C}$. Any inner face $\mathcal{F}^r$ is of the form $\mathcal{F}^r = \overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i}$, where $\mathcal{A}_j$ is of maximal dimension and $\mathcal{A}_i$ contains the interior of $\mathcal{F}^r$ [KKL14, Theorem 4.2(2)]. The index i is uniquely determined by this property. It follows from the proof of [HM13, Theorem 3.3(4)] that

$$\mathcal{F}^r = \{D \in \overline{\mathcal{A}_j} \mid (\varphi_j)_* D \cdot C = 0 \ \forall C \in N_1(Z_j/Z_i)\}$$

is in the vector space spanned by $\varphi_i^* \operatorname{Nef}(Z_i)$ and the exceptional locus of φ_j and $r = \rho(Z_j/Z_i)$.

Two types of codimension 1 faces: The following description of inner faces with codimension $r = 1$ follows from the description of the K-MMP with scaling on $\mathcal{C}$ and the above properties.

- Suppose that $\mathcal{F}^1 = \overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_k}$ for some distinct chambers of maximal dimension.
  1. If $\mathcal{A}_i$ is of maximal dimension, we can take $\mathcal{A}_k = \mathcal{A}_i$ and then $\varphi_{ji} \colon Z_j \longrightarrow Z_i$ is a divisorial contraction.
  2. If $\mathcal{A}_i$ is not of maximal dimension, then both φ_{ki} and φ_{ji} are small contractions and the induces birational map $Z_j \dashrightarrow Z_k$ is a log-flip.
- If $\mathcal{F}^1$ is contained in the closure of a unique chamber $\overline{\mathcal{A}_j}$ of maximal dimension, then $\mathcal{F}^1 \subseteq \partial^+\mathcal{C}$. In particular, $\dim Z_i < \dim Z$ and $\varphi_{ji} \colon Z_j \longrightarrow Z_i$ is a rank 1 fibration (i.e. terminal Mori fibre space). For this, see also [Kal13, Lemma 3.2].

We extend this description for inner codimension r faces:

Proposition II.B ([BLZ21, Proposition 4.25], [LZ20, Proposition 3.10])**.** *Let $\mathcal{F}^r \subseteq \partial^+\mathcal{C}$ be an inner face of codimension r and write $\mathcal{F}^r = \overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i}$ as above with $\mathcal{A}_j$ a chamber of maximal dimension and $\mathcal{A}_i \subseteq \partial^+\mathcal{C}$ the chamber containing the interior of $\mathcal{F}^r$. Then*

1. *the associated morphism $\varphi_{ji}\colon Z_j \longrightarrow Z_i$ is a rank r fibration;*
2. *if $\mathcal{F}^s \subseteq \partial^+\mathcal{C}$ is an inner codimension s face and $\mathcal{F}^r \subseteq \mathcal{F}^s$, then the rank r fibration φ_{ji} associated to $\mathcal{F}^r$ factorises through the rank s fibration associated to $\mathcal{F}^s$.*

The first part of the following Corollary II.C is [HM13, Theorem 3.7], and the second part is a natural generalisation.

Corollary II.C ([BLZ21, Corollary 4.27], [LZ20, Corollary 3.11, Corollary 3.13])**.**

1. *If the intersection $\mathcal{F}^1_i \cap \mathcal{F}^1_j$ of non-big codimension 1 inner faces has codimension 2, then there is a Sarkisov link between the corresponding Mori fibre spaces.*
2. *Let $\mathcal{F}^3 \subseteq \partial^+\mathcal{C}$ be an inner face of codimension 3 and T/B the associated rank 3 fibration from Proposition II.B. Then the elementary relation associated to T/B corresponds to the finite collection of codimension 1 faces $\mathcal{F}^1_1, \dots, \mathcal{F}^1_s$ containing $\mathcal{F}^3$, and ordered such that $\mathcal{F}^1_j$ and $\mathcal{F}^1_{j+1}$ share a codimension 2 face for all j (where the indexes are taken modulo s).*

Let us illustrate the Mori chambers $\mathcal{A}_i$ and the inner codimension r faces $\mathcal{F}^r$ on the blow-up of $\mathbb{P}^2$ in two distinct points.

Example II.2.2 ([Kal13, §1],[BLZ21, Examples 4.20&4.26])**.** We illustrate the definition of Mori chambers and faces on the example of the blow-up $Z \longrightarrow \mathbb{P}^2$ at two distinct points p_1 and p_2, see Figure II.2 and Figure II.4. We denote by $E_1, E_2 \subset Z$ the curves contracted onto $p_1, p_2 \in \mathbb{P}^2$ respectively, by L the strict transform of the line through p_1 and p_2, and by $H = L + E_1 + E_1$ the pull-back of a general line. The cone $\mathrm{Eff}(Z)$ of effective divisors on Z is the closed convex cone generated by E_1, E_2 and L, which are the only (-1)-curves on Z, while the cone $\mathrm{Nef}(Z)$ is the closed convex cone generated by $H, H - E_1$ and $H - E_2$. The anti-canonical divisor $-K_Z = 3H - E_1 - E_2 = 3L + 2E_1 + 2E_2$ is ample. In the figure we represent an affine section of the cone, and all divisors must be understood up to rescaling by an adequate homothety: for instance this is really $-\frac{1}{7}K_Z$ that is in the same affine section as E_1, E_2 and L, but for simplicity we write $-K_Z$.

There are eight Mori chambers $\mathcal{A}_0, \dots, \mathcal{A}_7$, see Figure II.3, corresponding to morphisms $\varphi_i\colon Z \longrightarrow Z_i$, $i = 0, \dots, 7$ to the varieties $Z_0 = Z$, $Z_1 = Z_2 = \mathbb{F}_1$, $Z_3 = \mathbb{F}_0$, $Z_4 = \mathbb{P}^2$, $Z_5 = Z_6 = \mathbb{P}^1$ and $Z_7 = \mathrm{pt}$ in the commutative diagram belog, φ_0 being the identity. The two morphisms $\varphi_{14}, \varphi_{24}\colon \mathbb{F}_1 \longrightarrow \mathbb{P}^2$ are the blow-ups of $p_1, p_2 \in \mathbb{P}^2$ respectively, and $\varphi_1, \varphi_2\colon Z \longrightarrow \mathbb{F}_1$ are the blow-ups of the images of p_1 and p_2. The morphisms $\varphi_{15}, \varphi_{26}\colon \mathbb{F}_1 \longrightarrow \mathbb{P}^1$ correspond to the $\mathbb{P}^1$-bundle of $\mathbb{F}_1$ and $\varphi_3 = \varphi_5 \times \varphi_6\colon Z \longrightarrow \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

The faces $\mathcal{F}^0_i = \overline{\mathcal{A}_i}$, $i = 0, \dots, 4$ are the faces of maximal dimension, and we write $\mathcal{F}^r_{ji} = \overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i}$, where r is its codimension. Every face of $\mathcal{C} = \mathrm{Eff}(Z)$ is inner.

The ample chamber $\mathcal{A}_0$ is the only open one and $\mathcal{A}_7$ is the only closed one. Moreover, as a hint that the behaviour of non maximal Mori chambers can be quite erratic, observe that $\mathcal{A}_7 = \overline{\mathcal{A}_7}$ is not connected, and that neither $\overline{\mathcal{A}_5}$ nor $\overline{\mathcal{A}_6}$ is a single face.

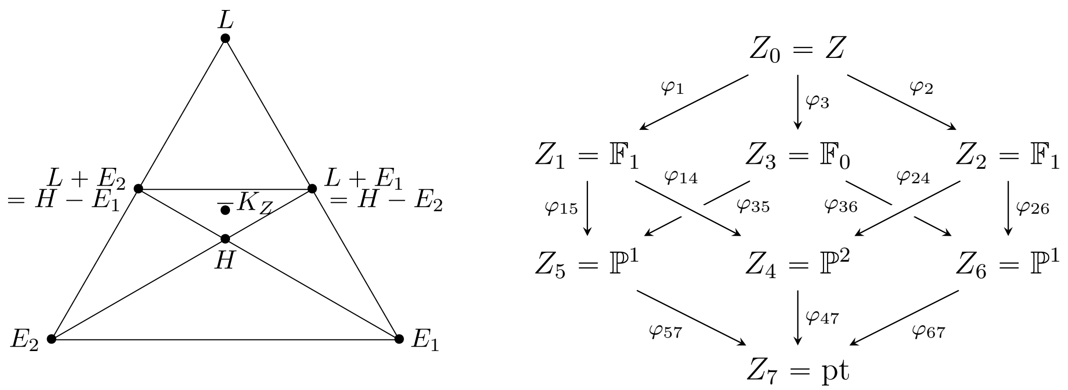


Figure II.2: The ample models in Example II.2.2.

$$
\begin{array}{lcl@{\qquad\qquad}lcl}
\mathcal{A}_0 & = & \mathring{\mathcal{F}}^0_0 & \mathcal{A}_4 & = & \mathring{\mathcal{F}}^0_4 \cup \mathring{\mathcal{F}}^1_{14} \cup \mathring{\mathcal{F}}^1_{24} \cup \mathring{\mathcal{F}}^2_{04} \\
\mathcal{A}_1 & = & \mathring{\mathcal{F}}^0_1 \cup \mathring{\mathcal{F}}^1_{01} & \mathcal{A}_5 & = & \mathring{\mathcal{F}}^1_{15} \cup \mathring{\mathcal{F}}^1_{35} \cup \mathring{\mathcal{F}}^2_{05} \\
\mathcal{A}_2 & = & \mathring{\mathcal{F}}^0_2 \cup \mathring{\mathcal{F}}^1_{02} & \mathcal{A}_6 & = & \mathring{\mathcal{F}}^1_{26} \cup \mathring{\mathcal{F}}^1_{36} \cup \mathring{\mathcal{F}}^2_{06} \\
\mathcal{A}_3 & = & \mathring{\mathcal{F}}^0_3 \cup \mathring{\mathcal{F}}^1_{03} & \mathcal{A}_7 & = & \mathring{\mathcal{F}}^1_{47} \cup \mathring{\mathcal{F}}^2_{17} \cup \mathring{\mathcal{F}}^2_{27} \cup \mathring{\mathcal{F}}^2_{37}
\end{array}
$$

Figure II.3: The Mori chambers and inner faces in Example II.2.2

[LZ20, Example 3.6] illustrates a similar example to the above but with two points in $\mathbb{P}^2$ of which one is infinitely near the other.

Example II.2.3. For rational Pezzo surface Z over $\mathbf{k} = \mathbb{Q}$ that is obtained by blowing up three points in $\mathbb{P}^2_{\mathbb{Q}}$, S. Lamy provides on his *webpage* a visualisation of $\mathcal{C}$, or, more precisely, of an affine section of $\mathcal{C}$.

II.2.3 Elementary relations generate all relations among links

Consider a composition of Sarkisov links

$$X_1/B_1 \overset{\chi_1}{\dashrightarrow} X_2/B_2 \overset{\chi_2}{\dashrightarrow} \cdots \dashrightarrow X_t/B_t \overset{\chi_t}{\dashrightarrow} X_{t+1}/B_{t+1}$$

between terminal Mori fibre spaces X_i/B_i, $i = 1, \dots, t+1$.

Compositions of links as paths on a 2-simplicial complex I: Take Z dominating the composition $\chi_t \circ \cdots \circ \chi_1$ and consider the cone $\mathcal{C}$ of Mori chambers. We can assume that $\rho(Z) \geqslant 4$; if it is not, we blow-up some general points on Z. This means that $\partial^+\mathcal{C}$ is a cone over a polyhedral complex of dimension $\rho(Z) - 2 \geqslant 2$ [BLZ21, Lemma 4.24] (see also [Kal13, Proposition 3.1]). In particular, a section S of $\partial^+\mathcal{C}$ is simply connected.

We consider the 2-skeleton of the dual cell complex of S, which is also simply connected. Its barycentric subdivision B is simply connected as well and it can be constructed as follows (up to homeomorphism): a vertex v_r corresponds to a face $\mathcal{F}^r$ of codimension $r \in \{1, 2, 3\}$, and two vertices v_r, v_s are joined if the corresponding face $\mathcal{F}^r$ is a proper face of $\mathcal{F}^s$. The subcomplex $I \subseteq B$ corresponding to inner faces of $\partial^+\mathcal{C}$ is a deformation

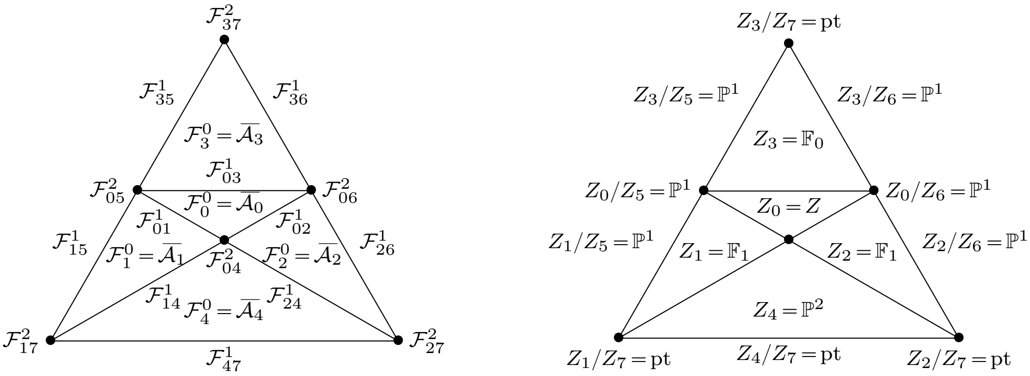


Figure II.4: The Mori chambers and rank r fibrations in Example II.2.2

retract of B, because the inner faces are the ones intersecting the relative interior of S. Hence I is simply connected as well.

By Proposition II.B, each vertex of I corresponds to an inner face in $\partial^+\mathcal{C}$ of codimension $r = 1, 2$ or 3, and they are connected by an edge if and only if the corresponding rank r fibrations factor through each other.

The composition $\chi_t \circ \cdots \circ \chi_1$ corresponds to a path in I through vertices of the form v_1 and v_2.

Relations of links are loops I: By Proposition II.B, each vertex in I of the form v_3 is the center of a disc whose boundary corresponds to the elementary relation of Sarkisov links dominated by the rank 3 fibration corresponding to v_3.

A relation $\chi_t \circ \cdots \circ \chi_1 = \mathrm{id}$ corresponds to a loop in I, and since I is simply connected, that loop is homotopic inside I to the constant loop. This means the loop can be filled up with the discs around vertices of the form v_3, which means that the relation $\chi_t \circ \cdots \circ \chi_1 = \mathrm{id}$ is the composition of conjugates of elementary relations.

We have shown that for any variety X of dimension $\geqslant 2$ over $\mathbb{C}$, the elementary relations in $\mathrm{BirMori}(X)$ generate all relations among Sarkisov links (and isomorphisms) in $\mathrm{BirMori}(X)$.

II.3 Elementary relations in dimension 2 over a perfect field

Throughout this section, $\mathbf{k}$ is a perfect field unless mentioned otherwise. The complete list of elementary relations of Sarkisov links between rational surfaces over a perfect field can be found in Appendix A. For $\mathbf{k} = \mathbb{C}$ and $\mathbf{k} = \mathbb{R}$, the lists have reasonable length as explained in the following two examples. An elementary relation referred to (X, a, b) means that the dominating rank 3 fibration is obtained by blowing up the del Pezzo surface X with $\rho(X) = 1$ in a point of degree a and a point of degree b.

Example II.3.1. A Mori fibre space birational to $\mathbb{P}^2_{\mathbb{C}}$ is isomorphic to $\mathbb{P}^2_{\mathbb{C}}$ or a Hirzebruch surfaces $\mathbb{F}_n/\mathbb{P}^1$, $n \geqslant 0$. Thus the elementary relations of Sarkisov links in $\mathrm{BirMori}(\mathbb{P}^2_{\mathbb{C}})$ are §A.2.6($\mathbb{P}^2, 1, 1$) and the relations in Remark A.1.1 involving links between Hirzebruch surfaces.

Example II.3.2 ([Zim18b, §2]). Any Mori fibre space birational to $\mathbb{P}^2_{\mathbb{R}}$ is isomorphic to $\mathbb{P}^2_{\mathbb{R}}$, the quadric surface $\mathcal{Q} = \{w^2 + x^2 + y^2 = z^2\} \subset \mathbb{P}^3$, a Hirzebruch surface $\mathbb{F}_n/\mathbb{P}^1$, $n \geqslant 0$, or the conic bundle $\mathcal{S}/\mathbb{P}^1$ obtained by blowing up $\mathcal{Q}$ in a pair of non-real conjugate points not contained in the same fibre of $\mathcal{Q}_{\mathbb{C}}$. (The fibres of $\mathcal{S}/\mathbb{P}^1$ are the strict transforms of the bidegree $(1,1)$-curves through the two blown-up points). Thus the elementary relations between Sarkisov links in $\mathrm{BirMori}(\mathbb{P}^2_{\mathbb{R}})$ are §A.2.6($\mathbb{P}^2, 1, 1$)&($\mathbb{P}^2, 1, 2$), and §A.2.5($\mathbb{P}^2, 2, 2$), and §A.2.4($\mathcal{Q}, 2, 2$), and the relations in Remark A.1.1 involving links between Hirzebruch surfaces and links from $\mathcal{S}/\mathbb{P}^1$ to itself. Theorem IV.A states that $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{R}})$ is a free product of two groups amalgamated along their intersection, and it is proven using the fact that these relations generate all relations in $\mathrm{BirMori}(\mathbb{P}^2_{\mathbb{R}})$.

The following example is a special trivial relation in $\mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}})$ over a perfect field.

Example II.3.3 ([LZ20, Lemma 4.3],[IKT93, §2]). Let X_1 be a rational del Pezzo surface with $\rho(X) = 1$, so that X_1/pt is a Mori fibre space. Suppose it contains a point p of degree $\deg(p) = K_{X_1}^2 - 1$ and that its blow-up $T \longrightarrow X_1$ yields a del Pezzo surface T, which is then of degree 1. Then T/pt is a rank 2 fibration and dominates a Sarkisov link $\chi\colon X_1 \dashrightarrow X_1$, called a *Bertini link*, which has base-locus p.

Geometrically, χ is defined as follows: there exists a unique rational point $q \in X_1$ such that for a general point $t \in X_1$ there is a unique smooth elliptic curve in X_1 through p, q, t. The map $t \mapsto -t$, where $-t$ is the opposite of t with respect to the group law on the elliptic curve, induces a birational involution $\beta\colon X_1 \dashrightarrow X_1$ not defined at p, called *Bertini involution*. There exists $\alpha \in \mathrm{Aut}(X_1)$ such that $\chi \circ \alpha = \beta$.

Since T is a del Pezzo surface of degree 1, a Bertini link is never dominated by a rank 3 fibration, and $\beta^2 = \mathrm{id}$ is a trivial relation.

Theorem II.3.4 below gives a short description of generating relations in $\mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}})$, and it was proven by J. Schneider independently of Theorem II.A or [IKT93] with a beautiful self-contained proof that studies linear systems of curves. The *Galois depth* of a birational map $\chi\colon X \dashrightarrow X'$ between smooth projective surfaces is the maximal degree among all base-points of χ and χ^{-1}.

Theorem II.3.4 ([Sch19, Theorem 2]). *Let* $\mathbf{k}$ *be a perfect field and* X *a projective surface over* $\mathbf{k}$. *Then the relations in* $\mathrm{BirMori}(X)$ *are generated by the trivial relations and relation of the form*

- $\chi_n \circ \cdots \circ \chi_1 = \mathrm{id}$, *where the Galois depth of all* χ_i *is* $\leqslant 15$, *and*
- $\chi_4 \circ \chi_3 \circ \chi_2 \circ \chi_1 = \mathrm{id}$, *where for* $i = 1, \ldots, 4$, χ_i *is a link of type* II *between Mori fibre spaces over a curve and* χ_3 *is not defined in the image by* χ_2 *of the base-point of* χ_1^{-1}.

Remark II.3.5. In Theorem II.3.4 the degree of the points blown up by the first type of relations is bounded by 15, which arises from the method used to prove Theorem II.3.4 in [Sch19]. Theorem II.3.4 can also be deduced from Theorem II.A(1), and then we can lower the bound of the Galois depth to 8: Let $\chi_1 \colon X_1/B_1 \dashrightarrow X_1'/B_1'$ be a Sarkisov link appearing in an elementary relation dominated by a rank 3 fibration $\eta_3 \colon X_3 \longrightarrow B_3$. We can assume that χ_1 that it is not an isomorphism. Let $\eta_2 \colon X_2 \longrightarrow B_2$ be the rank 2 fibration dominating χ_1. There is a biratonal morphism $\varphi \colon X_3 \longrightarrow X_2$ making the following diagram commute.

$$\begin{tikzcd} X_3 \arrow[r, "\varphi"'] \arrow[rrrrr, bend left, "\eta_3"] & X_2 \arrow[r, "\pi"] \arrow[rrr, bend right, "\eta_2"'] & X_1 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \end{tikzcd}$$

• Suppose that B_3 is a point. Then X_3 is a del Pezzo surface, and hence also X_2 and X_1 are del Pezzo surfaces. If B_1 is a point as well, then $\rho(X_3/X_1) = 2$ and χ_1 is a link of link of type I or a link of type II over a point. It follows that X_1' is a del Pezzo surface as well. In any case, the Galois depth of χ_1 is at most 8. If B_1 is a curve, then $\rho(B_1/B_3) = 1$, which implies that $\varphi \colon X_3 \longrightarrow X_2$ or $\pi \colon X_2 \longrightarrow X_1$ is an isomorphism, but not both. If π is an isomorphism, then χ_1 is a link of type III and X_1' is a del Pezzo surface as well, so the Galois depth of χ_1 is at most 8. If φ is an isomorphism, then χ_1 is a link of type II over B_1. Since such a link is symmetric and X_3, X_2, X_1 are del Pezzo surfaces, it follows that the Galois depth of χ_1 is at most 8.

• Suppose now that B_3 is a curve. Then $B_1 \longrightarrow B_2 \longrightarrow B_3$ is an isomorphism and $\pi \circ \varphi$ is a sequence of blow-ups of points contained in distinct smooth fibres over B_3 and whose geometric components are in distinct geometric fibres. This is the case for all Mori fibre space appearing as domain or target of a Sarkisov link in the elementary relation dominated by X_3/B_3, so all links appearing in this relation are links of type II between Mori fibre spaces over B_3. The only birational contractions starting from X_3 over B_3 are contractions of components of singular fibres. Since $\pi \circ \varphi$ is the blow-up of two points over B_3, the fibration X_3/B_3 has only two singular fibres whose components can be contracted from X_3 over $\mathbf{k}$. It follows that there are only four contractions starting from X_3 over B_3.

II.4 Elementary relations in dimension ⩾ 3

In this subsection, the base field is $\mathbb{C}$, except if mentioned otherwise. Elementary relations dominated by a terminal Fano variety of Picard rank 3 are studied in [Kal13, Example 4.9]. A beautiful elementary relation between Sarkisov links between non-rational Fano threefolds is presented in [AZ16, §5.2]. The following examples of relations among Sarkisov links are analogous to the relations explained in the previsous section.

II.4.1 Relations involving Bertini type links

Let X_1/B_1 and X_2/B_2 be Mori fibre spaces with $\dim X_1 = \dim X_2 = 3$. A *Bertini type link* is a Sarkisov link $\chi \colon X_1 \dashrightarrow X_2$ of type II over a curve $B_1 = B_2$ with base-locus a curve $\Gamma_1 \subset X_1$, such that the generic fibre of the rank 2 fibration dominating χ is a del Pezzo surface of degree 1. The inverse χ^{-1} is not defined in a curve birational to Γ_1

[BY20, Remark 2.7], and we denote by $g(\chi)$ the geometric genus of Γ_1.The elementary relations involving Bertini type links are studied in [BY20] using Theorem II.A(2) and Proposition II.4.2:

Proposition II.4.1 ([BY20, Proposition 3.3, Proposition 3.6])**.**

1. *There exists $g \geqslant 1$ such that no Bertini type link χ with $g(\chi) \geqslant g$ occurs in a non-trivial elementary relation dominated by a rank 3 fibration over a point.*
2. *If $\chi_1\colon X_1/B \dashrightarrow X_2/B$ is a Bertini type link appearing in a non-trivial elementary relation dominated by a rank 3 fibration over a curve, then this relation is of the form $\chi_4 \circ \chi_3 \circ \chi_2 \circ \chi_1 = \mathrm{id}$, where $\chi_3\colon X_1'/B' \dashrightarrow X_2'/B'$ is a Bertini type link, and χ_2 induces an isomorphism between the generic fibres of X_2/B and X_1'/B' and an isomorphism $B \longrightarrow B'$.*

II.4.2 Relations involving conic fibrations

A *Mori conic bundle* is a rank 1 fibration X/B with $\dim B = \dim X - 1$. Two Mori conic bundles X/B and X'/B' are *equivalent* if there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\psi"] \arrow[d] & X' \arrow[d] \\ B \arrow[r, dashed, "\theta"] & B' \end{tikzcd}$$

where θ, ψ are birational. A *marked Mori conic bundle* is a triple $(X/B, \Gamma)$, where X/B is a Mori conic bundle and its *marking* $\Gamma \subset B$ is an irreducible hypersurface not contained in the discriminant locus of X/B. Two marked Mori conic bundles $(X/B, \Gamma)$ and $(X'/B', \Gamma')$ are *equivalent* if there exists a commutative diagram as above such that the restriction of θ induces a birational map $\Gamma \dashrightarrow \Gamma'$ between the markings. Let $\chi\colon X_1/B \dashrightarrow X_2/B$ be a Sarkisov link of type II between of Mori conic bundles of dimension $n \geqslant 2$. Recall that χ fits in a commutative diagram of the form

$$\begin{tikzcd} Y_1 \arrow[rr, dotted, "\varphi"] \arrow[d, "\pi_1"'] & & Y_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[rr, dashed, "\chi"] \arrow[dr, "\eta_1"'] & & X_2 \arrow[dl, "\eta_2"] \\ & B & \end{tikzcd} \tag{II.1}$$

where $Y_1/B, Y_2/B$ are rank 2 fibrations, φ is a sequence of log-flips, each π_i is a divisorial contraction with exceptional divisor $E_i \subset Y_i$ and centre $\Gamma_i = \pi_i(E_i) \subset X_i$, and B is $\mathbb{Q}$-factorial and klt [Fuj99, Corollary 4.6].

Proposition II.D ([BLZ21, Lemma 2.13, Lemma 3.23])**.** *Let $\chi\colon X_1/B \dashrightarrow X_2/B$ be a Sarkisov link of type* II *between Mori conic bundles with $n := \dim X_1 \geqslant 3$. Then there exists an irreducible hypersurface $\Gamma \subset B$ (of dimension $n-2$) such that*

1. *for $i = 1, 2$, the centre $\Gamma_i = \pi_i(E_i)$ has codimension 2 in X_i, and the restriction $\eta_i|_{\Gamma_i}\colon \Gamma_i \longrightarrow \Gamma$ is birational. In particular, for each i we have $\eta_i \circ \pi_i(E_i) = \Gamma$, and the marked Mori conic bundles $(X_1/B, \Gamma)$ and $(X_2/B, \Gamma)$ are equivalent.*

2. *Let Y be equal to Y_1, Y_2, or any one of the intermediate varieties in the sequence φ of log-flips. Then $E_1 \cup E_2$ is the Zariski closure of the set of fibres of dimension 1 over Γ.*
3. *Γ is not contained in the discriminant locus of η_1, or equivalently of η_2, which means that a general fibre of $\eta_i \colon \eta_i^{-1}(\Gamma) \longrightarrow \Gamma$ is isomorphic to $\mathbb{P}^1$.*
4. *At a general point $x \in \Gamma_i$, the fibre of X_i/B through x is transverse to Γ_i.*
5. *For $i = 1, 2$, π_i is locally the blow-up of Γ_i.*

In dimension 2 over a perfect field, the analogous statement of Proposition II.D for a link of type II between Mori conic bundles is precisely the definition of such a link. The base-point p of the link and the base-point p' of its inverse are isomorphic $\mathbf{k}$-varieties of dimension 0. In particular, the minimal degree of extensions $\mathbf{k}(p)/\mathbf{k}$ and $\mathbf{k}(p')/\mathbf{k}$ are equal. We now generalise this for the hypersurface $\Gamma \subset B$ in the Proposition II.D.

For a curve C, one defines the *gonality* $\mathrm{gon}(C)$ of C to be the minimal possible degree of a dominant rational map $C \dashrightarrow \mathbb{P}^1$. We have $\mathrm{gon}(C) = 1$ if and only if C is rational, we have $\mathrm{gon}(C) = \deg(C) - 1$ if C is a smooth plane curve of degree $\geqslant 2$, and $\mathrm{gon}(C) = 2$ if C is a hyperelliptic curve.

More generally, we define the *covering gonality* and the *connecting gonality* of a variety Γ as in [BDE+17]:

1. The *covering gonality* $\mathrm{cov.gon}(\Gamma)$ of Γ is the minimal real number $c > 0$ such that there is an open dense set $U \subset \Gamma$ such that each point in U is contained in an irreducible curve $C \subset \Gamma$ with $\mathrm{gon}(C) \leqslant c$.
2. The *connecting gonality* $\mathrm{conn.gon}(\Gamma)$ of Γ to be the smallest real number $c > 0$ such that there is an open dense subset $U \subset \Gamma$ such that any two points in U are contained in an irreducible curve $C \subset \Gamma$ with $\mathrm{gon}(C) \leqslant c$.

If Γ is a closed subset of $\mathbb{P}^n$, we can project it onto a linear subspace of $\mathbb{P}^n$ of dimension $\dim(\Gamma)$. The preimages of general lines cover an open dense subset of Γ, so that $\mathrm{cov.gon}(\Gamma) \leqslant \deg(\Gamma)$. [BDE+17, Theorem A] states that if $\Gamma \subset \mathbb{P}^{n+1}$ is an irreducible hypersurface of degree $d \geqslant n+2$ with canonical singularities, then $\mathrm{cov.gon}(\Gamma) \geqslant d - n$. For a link χ of type II between Mori conic bundles as in Proposition II.D, we define

$$\mathrm{cov.gon}(\chi) := \mathrm{cov.gon}(\Gamma).$$

We associate to χ the equivalence class of the marked Mori conic bundle $(X_1/B, \Gamma)$. We say that two Sarkisov links of type II between Mori conic bundles are *equivalent* if their associated markings are equivalent. The following statement is the higher dimensional analogon to Theorem II.3.4.

Theorem II.E ([BLZ21, Proposition 5.3, Proposition 5.5])**.** *For each dimension $n \geqslant 3$, there exists an integer $d_n \geqslant 1$ depending only on n such that the following holds. If χ is a Sarkisov link of type* II *between Mori conic bundles that arises in an elementary relation dominated by a rank* 3 *fibration T/B with $\dim T = n$ then:*

1. *If $\dim B \leqslant n-2$, then $\mathrm{cov.gon}(\chi) \leqslant \max\{d_n, 8\,\mathrm{conn.gon}(T)\}$.*
2. *If $\dim B = n-1$ and $\mathrm{cov.gon}(\chi) > 1$, then the elementary relation has the form $\chi_4 \circ \chi_3 \circ \chi_2 \circ \chi = \mathrm{id}$, where χ_3 is a Sarkisov link of Mori conic bundles of type* II *equivalent to χ.*

An elementary relation as in Theorem II.E(2) has one of the three forms shown in Figure II.5, where the varieties are organized in circles according to their Picard rank over B [BLZ21, Proof of Proposition 5.5].

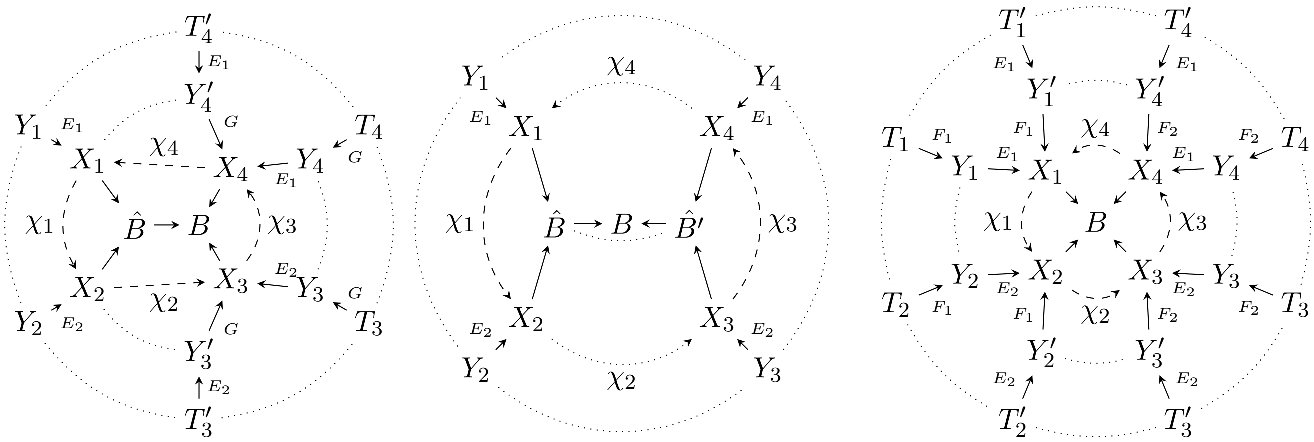


Figure II.5: The elementary relations associated to T/B in Theorem II.E(2), where E_i, F_i and G are the divisors contracted by the corresponding arrow. In the center, B is not $\mathbb{Q}$-factorial, and B is $\mathbb{Q}$-factorial in the other two cases.

About the constant d_n in Theorem II.E

The constant d_n in Theorem II.E comes from a consequence (Proposition II.4.2) of the Borisov-Alexeev-Borisov conjecture, which was proven by C. Birkar in any dimension.

Proposition II.4.2 ([BLZ21, Proposition 5.1])**.** *Let n be an integer and $\mathcal{Q}$ the set of weak Fano terminal varieties of dimension n. There are integers $d, l, m \geqslant 1$, depending only on n, such that for each $X \in \mathcal{Q}$, the following hold:*

1. $\dim H^0(-mK_X) \leqslant l$;
2. *the linear system $|-mK_X|$ is base-point free;*
3. *the morphism $\varphi\colon X \xrightarrow{|-mK_X|} \mathbb{P}^{h^0(-mK_X)-1}$ is birational onto its image and contracts only curves $C \subseteq X$ with $C \cdot K_X = 0$;*
4. $\deg \varphi(X) \leqslant d$.

Proposition II.4.2 is essentially assembled from [Bir21, Bir19, Kol93]. The d_n in Theorem II.E is the maximum of the d's from Proposition II.4.2 for dimension $1, 2\ldots, n$. Let us explain this roughly. Let $\chi\colon X_1 \dashrightarrow X_2$ be a link of type II appearing in an elementary relation dominanted by a rank 3 fibration T/B, where $X_1/\tilde{B}$, $X_2/\tilde{B}$ are Mori conic bundles. Consider the case $\dim B < \dim \tilde{B} = n-1$. If $\Gamma_1 \subset X_1$ is the base-locus of χ, we need to show that $\operatorname{cov.gon}(\Gamma_1) \leqslant \max\{d_n, 8\operatorname{conn.gon}(T)\}$. Let us assume that $\operatorname{cov.gon}(\chi) > 1$.

Go to nicer rank 2- and 3 fibrations: In the diagram below, π_1, π_2 are the birational morphisms from (II.1), where π_1 is locally the blow-up of Γ_1 by Proposition II.D. The conditions on the relative Picard ranks imply that the birational contractions $T \dashrightarrow Y_i$, $i = 1, 2$, are pseudo-isomorphisms. Then $\tilde{B}/B$ is a klt Mori fibre space [BLZ21, Lemma 3.13] and X_1/B is a rank 2 fibration [BLZ21, Lemma 3.4(1)]. By Remark II.1.6 there is a pseudo-isomorphism $X_1 \dashrightarrow X$ over B to a rank 2 fibration X/B with $-K_X$ relatively big and nef. Since $Y_1 \longrightarrow X_1$ is locally a blow-up of Γ, $X_1 \dashrightarrow X$ lifts to a pseudo-isomorphism $Y_1 \dashrightarrow Y$ over B to a rank 3 fibration Y/B [BLZ21, Lemma 2.17]. Again by Remark II.1.6 there is pseudo-isomorphism $Y \dashrightarrow \hat{Y}$ over B to a rank 3 fibration $\hat{Y}/B$ with $-K_{\hat{Y}}$ relatively big and nef.

$$\begin{tikzcd}
\hat{Y} \arrow[r, dotted, no head] \arrow[dddrrr, bend right] & Y \arrow[r, dotted, no head] \arrow[d] & Y_1 \arrow[r, dotted, no head] \arrow[d, "\pi_1"] & T \arrow[r, dotted, no head] & Y_2 \arrow[d, "\pi_2"] \\
 & X \arrow[r, dotted, no head] \arrow[ddrr] & X_1 \arrow[rr, dashed, "\chi"] \arrow[dr] & & X_2 \arrow[dl] \\
 & & & \tilde{B} \arrow[d] & \\
 & & & B &
\end{tikzcd}$$

Let $\Gamma \subset X$ be the image of $\Gamma_1 \subset X_1$. The induced map $\Gamma_1 \dashrightarrow \Gamma$ is birational, so it suffices to show that $\operatorname{cov.gon}(\Gamma) \leqslant d_n$. The morphism $\Gamma \longrightarrow B$ is surjective, because otherwise the image of Γ would be a divisor, which is impossible because fibres of $\tilde{B}/B$ are covered by rational curves [HM07, Corollary 1.5(1)] and $\operatorname{cov.gon}(\Gamma) = \operatorname{cov.gon}(\Gamma_1) > 1$.

Look at the fibre above a general point: Let X_p, Y_p, $\hat{Y}_p$ be the fibres above a general point $p \in B$. Then X_p and $\hat{Y}_p$ are weak terminal Fano variety of dimension $n_0 := n - \dim B \in \{2, \dots, n\}$ [Kol97, 7.7]. Then $\Gamma_p := \Gamma \cap X_p$ is the fibre of $\Gamma \longrightarrow B$ above p, and the restriction $Y_p \longrightarrow X_p$ is the blow-up in Γ_p.

If $n_0 = 2$, then $Y_p \simeq \hat{Y}_p$ is a del Pezzo surface and so Γ_p is the union of at most 8 points. We obtain $\operatorname{cov.gon}(\Gamma) \leqslant 8 \operatorname{cov.gon}(B) \leqslant 8 \operatorname{conn.gon}(T)$ [BLZ21, Lemma 2.22(3)].

If $n_0 \geqslant 3$, we consider the birational morphisms $\varphi_p \colon X_p \longrightarrow \mathbb{P}^{h^0(-mK_{X_p})-1} = \mathbb{P}^a$ and $\hat{Y}_p \longrightarrow \mathbb{P}^{h^0(-mK_{\hat{Y}_p})-1} = \mathbb{P}^b$ from Proposition II.4.2. They are pseudo-isomorphisms because the locus covered by curves with non-positive intersection against the anticanonical divisor is of codimension $\geqslant 2$ by Remark II.1.6. Moreover, $b \leqslant a$, because $Y_p \longrightarrow X_p$ is the blow-up in Γ_p. So, Γ_p is not contained in the exceptional locus of φ_p, and φ_p induces a birational morphism $\Gamma_p \longrightarrow \varphi_p(\Gamma_p)$. We have a commutative diagram

$$\begin{tikzcd}
X_p \arrow[d, "|-mK_{X_p}|"', "\varphi_p"] & Y_p \arrow[l] \arrow[rr, dotted, no head] \arrow[dr, dashed, "|-mK_{Y_p}|"'] & & \hat{Y}_p \arrow[dl, "|-mK_{\hat{Y}_p}|"] \\
\mathbb{P}^a \arrow[rr, dashed, "\pi_p"'] & & \mathbb{P}^b &
\end{tikzcd}$$

where π_p is the projection away from a linear subspace $\mathcal{L}$ containing $\varphi_p(\Gamma_p)$. One shows that all irreducible components of $\varphi_p(X_p) \cap \mathcal{L}$ are of dimension $\leqslant n_0 - 2$, and that the ones of dimension $n_0 - 2$ have degree $\leqslant d_{n_0}$, because $\varphi_p(X_p)$ is of degree $\leqslant d_{n_0}$ by Proposition II.4.2. Consider a general linear projection from $\mathbb{P}^a$ to a linear subspace of dimension $n_0 - 2$. For any $q \in \Gamma_p \backslash \operatorname{Ex}(\varphi_p)$ the preimage of a line through $\varphi_p(q)$ is a curve of gonality $\leqslant d_{n_0}$. We have now shown that for a general $p \in B$ and any point q in the open set $\Gamma_p \backslash \operatorname{Ex}(\varphi_p) \subset \Gamma$ there is a curve in Γ_p through q of gonality $\leqslant d_n$.

Γ is of covering gonality $\leqslant d_n$: We consider the rational map $| - mK_X| \times \eta \colon X \dashrightarrow$

$\mathbb{P}^N \times B$ with m as in Proposition II.4.2 for the general fibres of $\eta\colon X \longrightarrow B$, whose restriction to X_p is φ_p. We have found an open subset $U \subseteq \Gamma \setminus \mathrm{Ex}(\varphi)$ that is covered by curves of gonality$\leqslant d_n$.

III Homomorphisms from and of Cremona groups

When we study a group G we want to know, for instance, whether we can make the group simpler and perhaps even reduce the study of G to that simpler group. This means we want to know whether there exists a non-trivial normal subgroup $H \subsetneq G$. Equivalently, we want to know whether there exist any surjective non-trivial homomorphism from G to another group. A group that does not have any non-trivial proper normal subgroups is called *simple*, since there is no normal subgroup to quotient out in order to get a "simpler" group. The earliest written evidence (we know of) mentioning this problem for Cremona groups is in a book by F. Enriques from 1895:

> *Tuttavia altre questioni d'indole gruppale relative al gruppo Cremona nel piano (ed a più forte ragione in S_n $n > 2$) rimangono ancora insolute; ad esempio l'importante questione se il gruppo Cremona contenga alcun sottogruppo invariante (questione alla quale sembra probabile si debba rispondere negativamente). [Enr95, p. 116]*[1]

Let us elaborate on the known answers in dimension 2 and dimension $\geqslant 3$.

III.1 Homomorphism from the plane Cremona group

III.1.1 Over an algebraically closed field

Here is a reason why the plane Cremona group over an algebraically closed field $\mathbf{k}$ is not an easy group when it comes to normal subgroups. The Noether-Castelnuovo theorem [Cas01] states that $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is generated by $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ and the involution $\sigma\colon [x:y:z] \mapsto [yz:xz:xy]$. The smallest normal subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ containing any of the generators is in fact the whole group. Indeed, consider the involution $h\colon [x:y:z] \mapsto [x-z:y-z:z]$. Then $(\sigma h)^3 = \mathrm{id}$, so that $h = (h\sigma h)\sigma(h\sigma h)$ is a conjugate of σ. Moreover, $\mathrm{PGL}_3(\mathbf{k})$ is a simple group, so it follows that the smallest normal subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ containing σ (resp. any nontrivial element of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$) also contains $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ (resp. σ) and hence is equal to the whole of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$.

The non-simplicity of the plane Cremona group was only proven less than a decade ago. It was shown over $\mathbf{k} = \mathbb{C}$ in [CL13] and generalised over any field in [Lon16].

[1] "However, other group-theoretic questions related to the Cremona group of the plane (and, even more so, of $\mathbb{P}^n$, $n > 2$) remain unsolved; for example, the important question of whether the Cremona group contains any normal subgroup (a question which seems likely to be answered negatively)."

Theorem III.1.1 ([CL13, Lon16]). *Let* $\mathbf{k}$ *be an arbitrary field. Then* $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ *is not simple.*

The normal subgroups $N \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ constructed in [CL13, Lon16] are large, but also their quotients are large. For instance, the group of elements of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ preserving the pencil of lines through $[0:1:0]$ embeds into $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})/N$.

For an algebraically closed field $\mathbf{k}$, the fact that the smallest normal subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ containing any nontrivial element of $\{\sigma\} \cup \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$ is the whole group, implies that $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ does not have any finite quotients.

III.1.2 Over a non-closed perfect field

Contrary to the situation over algebraically closed fields, there are lots of finite quotients of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ if $\mathbf{k}$ is non-closed.

Given a group $G = \langle S \mid R\rangle$ and some group G', we can for each $s \in S$ define some element $\varphi(s) \in G'$ along with $\varphi(1) := 1$. If $\varphi(r) = 1$ for any $r \in R$, then $\varphi\colon G \longrightarrow G'$ is a homomorphism. This can be transferred to groupoids. The following homomorphisms from $\mathrm{BirMori}(X)$ or from $\mathrm{Bir}(X)$ are all constructed with this recipe by using the generators and generating relations of $\mathrm{BirMori}(X)$ by Sarkisov links and elementary relations from Theorem II.A or Theorem II.3.4.

Theorem III.1.2 ([Zim18a, Theorem 1.1], [Zim18b, Theorem 1.3]). *There exists a surjective homomorphism* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{R}}) \longrightarrow \bigoplus_I \mathbb{Z}/2$*, where* I *is uncountable. Its kernel is the derived subgroup of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{R}})$*, which is also the smallest normal subgroup containing* $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{R}})$.

Theorem III.1.2 is proven in [Zim18a] with much dirty work with relations inside $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{R}})$, and it is re-proven more compactly in [Zim18b] by using the list of elementary relations in $\mathrm{BirMori}(\mathbb{P}^2_{\mathbb{R}})$ from Example II.3.2.

Recall from Example II.3.3 that over any perfect field $\mathbf{k}$, Bertini involutions of $\mathbb{P}^2_{\mathbf{k}}$ do not appear in any non-trivial relation in $\mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}})$.

Theorem III.A ([LZ20, Theorem C(2)&(3)]). *Let* $\mathbf{k}$ *be a perfect field with a Galois extension of degree* 8 *and let* $\mathcal{B} \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ *a set of representatives of Bertini involutions of* $\mathbb{P}^2$ *up to conjugacy with* $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$*. Then* $|\mathcal{B}| \geqslant |\mathbf{k}|$ *and there is a surjective homomorphism*

$$\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow \mathop{\ast}_{\mathcal{B}} \mathbb{Z}/2,$$

which sends each $b \in \mathcal{B}$ *onto the corresponding generator on the right-hand side. In particular, the abelianisation of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ *contains a subgroup isomorphic to* $\bigoplus_{\mathcal{B}} \mathbb{Z}/2$.

Definition III.1.3. For a terminal Mori fibre space X/B, we define $\mathrm{CB}(X)$ to be the set of equivalence classes of Mori conic bundles birational to X (see §II.4.2). For a given class $C \in \mathrm{CB}(X)$, we define $M(C)$ to be the set of equivalence classes of Sarkisov links of type II between marked Mori fibre conic bundles $(Y/B, \Gamma)$ with C the class of Y/B (see §II.4.2). By $\mathrm{Bir}(X/B) \subset \mathrm{Bir}(X)$ we denote the subgroups of elements that preserve the fibration X/B.

Theorem II.3.4 (or Theorem II.A(1)) give rise to the following quotients of $\mathrm{BirMori}(X)$, $\mathrm{Bir}(X)$ and $\mathrm{Bir}(X/B)$.

Theorem III.1.4 ([Sch19, Theorem 3]). *Let X be projective surface over a perfect field $\mathbf{k}$. There exist a groupoid homomorphism*

$$\mathrm{BirMori}(X) \longrightarrow \mathop{\ast}_{C \in \mathrm{CB}(X)} \bigoplus_{\chi \in M(C)} \mathbb{Z}/2$$

that sends each Sarkisov link χ of type II *between Mori conic bundle with* $\operatorname{cov.gon}(\chi) \geqslant 16$ *onto the generator indexed by its equivalence class, and all other Sarkisov links and all automorphisms of Mori fibre spaces birational to X onto zero. Moreover, it restricts to group homomorphisms*

$$\mathrm{Bir}(X) \longrightarrow \mathop{\ast}_{C \in \mathrm{CB}(X)} \bigoplus_{\chi \in M(C)} \mathbb{Z}/2, \qquad \mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\chi \in M(X/B)} \mathbb{Z}/2.$$

Definition III.1.5. Let $\mathbf{k}$ be a perfect field. We denote by $\mathcal{S}$ a del Pezzo surface of degree 6 obtained by blowing up $\mathbb{P}^2$ in two points of degree 2 and then contracting the strict transform of the line through one of them. Let $\mathcal{X}$ be the del Pezzo surface of degree 5 obtained by blowing up a point in $\mathbb{P}^2$ of degree 4. They both carry a conic bundle structure, whose fibres are the strict transforms of the conics through the blown-up points.

Lemma III.1.6 ([Sch19, Proposition 6.11, Proposition 6.12]). *Let $\mathbf{k}$ be a perfect field and $X/\mathbb{P}^1$ a smooth rational Mori conic bundle over $\mathbf{k}$. Then X is isomorphic to a Hirzebruch surface, to some $\mathcal{S}$ or to some $\mathcal{X}$. Moreover, if $Y/\mathbb{P}^1$ is a smooth Mori conic bundle and $Y \dashrightarrow X$ a birational map to $X = \mathcal{S}$ or $X = \mathcal{X}$, then Y is isomorphic to X, and it is constructed by blowing up the same points in $\mathbb{P}^2$, up to* $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2)$.

In particular, the elements of $\mathrm{CB}(\mathbb{P}^2)$ are the classes of $\mathbb{F}_1/\mathbb{P}^1$, $\mathcal{S}/\mathbb{P}^1$ and $\mathcal{X}/\mathbb{P}^1$. Let J_6 (resp. J_5) parametrise the isomorphism classes of the $\mathcal{S}/\mathbb{P}^1$ (resp. $\mathcal{X}/\mathbb{P}^1$). The following quotient is obtained by composing the quotient in Theorem III.1.4 with a suitable projection.

Theorem III.1.7 ([Sch19, Theorem 4]). *Let $\mathbf{k}$ be a perfect field with $[\bar{\mathbf{k}} : \mathbf{k}] > 2$. There exists a surjective homomorphism*

$$\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow \bigoplus_{I_0} \mathbb{Z}/2 \ast (\mathop{\ast}_{J_6} \bigoplus_{I} \mathbb{Z}/2) \ast (\mathop{\ast}_{J_5} \bigoplus_{I} \mathbb{Z}/2)$$

where $I_0 \subset M(\mathbb{F}_1)$ is infinite and I is at least as big as the set of irreducible polynomials in $\mathbf{k}[x]$ of odd degree. In particular, there is a surjective homomorphisms $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{Q}}) \longrightarrow \ast_{\mathbb{N}} \mathbb{Z}/2$ and for any finite field $\mathbf{k}$ there is a surjective homomorphism $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \longrightarrow \mathbb{Z}/2 \ast \mathbb{Z}/2 \ast \mathbb{Z}/2$.

We dont know the kernel of this homomorphism, but for instance, by construction it contains all Bertini type links of $\mathbb{P}^2$.

III.1.3 The homomorphism given by factorisation centers

It is not so easy to construct explicitly a non-trivial homomorphism starting from the plane Cremona group. An intuitive idea is the following. Let $\mathbf{k}$ be a perfect field and $\varphi\colon X \dashrightarrow Y$ a birational map of smooth projective surfaces over $\mathbf{k}$. We have a decomposition

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\varphi"] & & Y \end{tikzcd}$$

with α and β compositions of blow-ups of closed points $p_1, \dots, p_r$ and $p'_1, \dots, p'_s$, respectively. We define $\mathrm{Var}^0/\mathbf{k}$ to be the set of isomorphism classes of irreducible zero-dimensional varieties over $\mathbf{k}$, and $\mathbb{Z}[\mathrm{Var}^0/\mathbf{k}]$ to be the $\mathbb{Z}$-module generated by $\mathrm{Var}^0/\mathbf{k}$. The ring $\mathbb{Z}[\mathrm{Var}^0/\mathbf{k}]$ is isomorphic to the Burnside ring of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$. The *factorisation center of* φ is defined as

$$c(\varphi) := \sum_{i=1}^{r}[p_i] - \sum[p'_i] \in \mathbb{Z}[\mathrm{Var}^0/\mathbf{k}]$$

It does not depend on the choice of factorisation of φ and defines a homomorphism c from the groupoid of birational maps between smooth projective surfaces $\mathrm{Bir}_2/\mathbf{k}$ to $\mathbb{Z}[\mathrm{Var}^0/\mathbf{k}]$ [LSZ20, Lemma 3.1]. Moreover, it restricts to a homomorphism of groups

$$c\colon \mathrm{Bir}(X) \longrightarrow \mathbb{Z}[\mathrm{Var}^0/\mathbf{k}].$$

If Z is a finite union of closed points in a smooth surface X, and $Y \longrightarrow X$ its blow-up, the action of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ on $Z_{\overline{\mathbf{k}}}$ induces a permutation representation of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ on $\mathrm{Pic}(Y_{\overline{\mathbf{k}}}) \otimes \mathbb{Q}$. The following example shows that this permutation representation does not determine the isomorphism class of Z.

Example III.1.8 ([LSZ20, Example 2.14], based on [Par13, §1.1])**.** Let $\mathbf{k} = \mathbb{Q}$ and let Z be the union of the three points $[\pm\sqrt{\alpha} : 1 : 0], [0 : \pm\sqrt{\beta} : 1], [1 : 0 : \pm\sqrt{\alpha\beta}]$ in $\mathbb{P}^2_{\mathbb{Q}}$ and Z' union of the points $[\pm\sqrt{\alpha} : \pm\sqrt{\beta} : 1], [1 : 0 : 1], [0 : 1 : 1]$ in $\mathbb{P}^2_{\mathbb{Q}}$. Let Y and Y' be the blow-up of Z and Z', respectively. They are not isomorphic, because Y contains only three $\mathbf{k}$-rational (-1)-curves and Y' contains five. On the other hand, we have $\mathrm{Pic}(Y_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} \simeq \mathrm{Pic}(Y'_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ as permutation representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

However, if Z is of degree $\leqslant 5$, or if Z is irreducible and of degree $\leqslant 6$, or if Z is irreducible and its splitting field is cyclic or Galois over $\mathbf{k}$, then permutation presentation $\mathrm{Pic}(Y_{\overline{\mathbf{k}}}) \otimes \mathbb{Q}$ of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ determines the isomorphism class of Z [LSZ20, Corollary 2.13].

The following statement is immediate if $\mathbf{k}$ is algebraically closed, $\mathbf{k} = \mathbb{R}$ or $\mathbf{k}$ is finite.

Theorem III.B ([LSZ20, Theorem 3.4])**.** *Let $\mathbf{k}$ be a perfect field. For any two smooth projective surfaces X, Y over $\mathbf{k}$, any two birational isomorphism $\varphi, \psi\colon X \dashrightarrow Y$ have $c(\varphi) = c(\psi)$. In particular, the homomorphism $c\colon \mathrm{Bir}(X) \longrightarrow \mathbb{Z}[\mathrm{Var}^0/\mathbf{k}]$ is trivial.*

For $\mathbf{k} = \mathbb{R}$ the second statement also frollows from Theorem III.1.2, because it implies that any homomorphism $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{R}}) \longrightarrow \mathbb{Z}$ must be trivial.

To show Theorem III.B for geometrically irreducible surfaces, it suffices to study birational maps $\varphi\colon X \dashrightarrow X'$ between outputs of K-MMPs. If K_X is nef, then $K_{X'}$ is nef and φ is an automorphism [IS96, Corollary 1 in II.7.3], so $c(\varphi) = 0$. If X, X' are Mori fibre spaces that are geometrically non-rational, then $c(\varphi) = 0$ [Sch19, Lemma 3.3]. If X is a minimal del Pezzo surface of degree $\leqslant 4$, it follows from the classification of Sarkisov links in [Isk96, Theorem 2.6] that $c(\varphi) = 0$.

For del Pezzo surfaces X_d of degree $d \geqslant 5$ and Hirzebruch surfaces, we define elements of $\mathbb{Z}[\mathrm{Var}^0/\mathbf{k}]$ as follows, where Z_i has i geometric components but is not necessarily irreducible.

- $A_{\mathbb{F}_n} := 2[\mathrm{Spec}(\mathbf{k})]$, $n \geqslant 1$,
- $A_{X_9} := [\mathrm{Spec}(\mathbf{k})]$,
- $A_{X_8} := [Z_2]$ where Z_2 parametrises the rulings of $(X_8)_{\overline{\mathbf{k}}}$ and X_8 is minimal,
- $A_{X_6} := [Z_2] + [Z_3] - [\mathrm{Spec}(\mathbf{k})]$, where Z_3 parametrises the three pencils of conics on $(X_6)_{\overline{\mathbf{k}}}$, and Z_2 parametrises two families on $(X_6)_{\overline{\mathbf{k}}}$: the strict transforms of general lines in $\mathbb{P}^2_{\overline{\mathbf{k}}}$ and of the conics in $\mathbb{P}^2_{\overline{\mathbf{k}}}$ through three points,
- $A_{X_5} := [Z_5]$, where Z_5 parametrises the five pencils of conics in $(X_5)_{\overline{\mathbf{k}}}$.

Proposition III.C ([LSZ20, Proposition 5.5])**.** *Let X be a del Pezzo surface of degree $\geqslant 5$ or a Hirzebruch surface, and $\varphi\colon X \dashrightarrow X'$ a birational map to the outcome of some K-MMP. Then X' is a del Pezzo surface of degree $\geqslant 5$ or a Hirzebruch surface, and $c(\varphi) = A_{X'} - A_X$.*

Proposition III.C concludes the proof of Theorem III.B for geometrically irreducible surfaces. A geometrically reducible surface X can be viewed as geometrically irreducible over the field of regular functions of X, which is a finite extension of $\mathbf{k}$.

The following corollary from Theorem III.B and Proposition III.C tells us that for any rational smooth projective surface X, any birational maps $\mathbb{P}^2 \dashrightarrow X$ must factor through the blow-ups or blow-downs of some special points associated to X.

Corollary III.D ([LSZ20, Corollary 5.9])**.** *There is a unique map*

$$\{\textit{Isomorphism classes of rational smooth projective surfaces}\} \xrightarrow{\mathcal{M}} \mathbb{Z}[\mathrm{Var}^0/\mathbf{k}]$$

such that for any birational map $\varphi\colon \mathbb{P}^2 \dashrightarrow X$ we have $\mathcal{M}(X) := c(\varphi) + 1$. We have $\mathcal{M}(X) = A_X$ for a minimal del Pezzo surface X of degree $\geqslant 5$ and for $X = \mathbb{F}_n$, $n \geqslant 1$.

E. Shinder and H.-Y. Lin have announced that the analogously defined homomorphism starting from $\mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$ is trivial as well, but that for the homomorphism starting from $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ is non-trivial if $n \geqslant 4$, as well as for the homomorphism starting from $\mathrm{Bir}(\mathbb{P}^3_{\mathbf{k}})$ over some non-closed field $\mathbf{k}$ [LS].

III.2 Homomorphisms from Cremona groups in dimension $\geqslant 3$

We are working over the field $\mathbb{C}$. The following results establish the existence of non-trivial homomorphisms of groups starting from $\mathrm{Bir}(X)$ for some large families of varieties

X. Some results can be formulated over subfields $\mathbf{k}$ of $\mathbb{C}$, simply by restricting the homomorphism to the subgroup of birational maps already defined over $\mathbf{k}$.

III.2.1 Using elementary relations among Bertini type links

Let X_1/B be a Mori fibre space with $\dim X_1 = 3$ and B a curve. A type link $\chi\colon X_1 \dashrightarrow X_2$ over B and a Bertini type link $\chi'\colon X_1' \dashrightarrow X_2'$ over B' are equivalent, if there are birational maps $\psi\colon X_1 \dashrightarrow X_1'$ and $\psi'\colon X_2 \dashrightarrow X_2'$ such that $\psi' \circ \chi = \chi' \circ \psi$ and that induce the same isomorphism $B \longrightarrow B'$. The description of the elementary relations among Bertini type links from Proposition II.4.1 yield the following generalisation of Proposition III.A to dimension 3.

Theorem III.2.1 ([BY20, Theorem D]). *There exists an integer $g \geqslant 0$ such that for any rank 1 fibration X/B of $\dim X = 3$ over a curve B, there exists a group homomorphism*

$$\mathrm{Bir}(X) \longrightarrow \mathop{\ast}_{I} \mathbb{Z}/2$$

which is the restriction of a groupoid homomorphism $\mathrm{BirMori}(X) \longrightarrow \ast_I \mathbb{Z}/2$ that sends every Sarkisov link χ of Bertini type with $g(\chi) \geqslant g$ (see §II.4.1 for the definition of $g(\chi)$) to the generator indexed by its equivalence class, and all other Sarkisov links and all automorphism of Mori fibre spaces birational to X onto the trivial element.

Theorem III.2.1 applies in particular to any threefold del Pezzo fibration X/B of degree 3 above a curve, and J. Blanc and E. Yasinsky show that after cutting off some factors, the induced homomorphism $\mathrm{Bir}(X) \longrightarrow \ast_{\mathbb{N}} \mathbb{Z}/2$ is surjective [BY20, Theorem A].

III.2.2 Using elementary relations among Mori conic bundles

Recall from Definition III.1.3 that $\mathrm{CB}(X)$ denotes the set of equivalence classes of Mori conic bundles and for $C \in \mathrm{CB}(X)$, we denote by $M(C)$ the set of equivalence classes of Sarkisov links of type II between marked Mori conic bundles.

We have the following higher dimensional analogues of Theorem III.1.4 and Theorem III.1.7.

Theorem III.E ([BLZ21, Theorem D]). *Let $n \geqslant 3$. There exists an integer $d_n > 1$ depending only on n such that for every terminal Mori conic bundle X/B of dimension n, we have a groupoid homomorphism*

$$\mathrm{BirMori}(X) \longrightarrow \mathop{\ast}_{C \in \mathrm{CB}(X)} \left(\bigoplus_{\chi \in M(C)} \mathbb{Z}/2 \right)$$

that sends each Sarkiso link of Mori conic bundles χ of type II *with $\mathrm{cov.gon}(\chi) > \max\{d_n, 8\,\mathrm{conn.gon}(X)\}$ onto the generator indexed by its equivalence class, and all other Sarkisov links and all automorphisms of Mori fibre spaces birational to X onto zero. More-*

over, it restricts to group homomorphisms

$$\mathrm{Bir}(X) \longrightarrow \ast \left(\bigoplus_{C \in M(C)} \mathbb{Z}/2 \right), \quad \mathrm{Bir}(X/B) \longrightarrow \bigoplus_{M(X/B)} \mathbb{Z}/2.$$

The homomorphisms in the following statements are compositions of the homomorphisms in Theorem III.E with a suitable projection.

The following example and proposition show that if X is birational to a special conic fibration, the above homomorphism is non-trivial and its image is large.

Example III.2.2 ([BLZ21, Lemma 6.5])**.** Let B be a smooth variety of dimension at least 2, $X = \mathbb{P}^1 \times B$, and let $\varphi_M \in \mathrm{Bir}(X/B) \simeq \mathrm{PGL}_2(\mathbb{C}(B))$ be the birational map

$$\varphi_M : ([u:v], t) \dashrightarrow ([a(t)u + b(t)v : c(t)u + d(t)v], t), \qquad ad - bc \neq 0.$$

Then the image of φ_M under the group homomorphism $\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{M(X/B)} \mathbb{Z}/2$ of Theorem III.E is equal to the sum of the equivalence classes of marked conic bundles $(X/B, \Gamma)$ such that $\Gamma \subset B$ is an irreducible hypersurface of B with $\operatorname{cov.gon}(\Gamma) > \max\{d, 8 \operatorname{conn.gon}(X)\}$ and such that the multiplicity of $ad - bc$ along Γ is odd.

Definition III.2.3. A *decomposable* $\mathbb{P}^2$*-bundle* P is the projectivisation of a decomposable rank 3 vector bundle, i.e. $P = \mathbb{P}(\mathcal{O}_{\mathbb{P}^m} \oplus \mathcal{O}_{\mathbb{P}^m}(a) \oplus \mathcal{O}_{\mathbb{P}^m}(b))$ for some $a, b \in \mathbb{Z}$. A Mori conic bundle X/B is called *decomposable* if we have closed embeddings $B \hookrightarrow \mathbb{P}^m$ and $X \hookrightarrow P$, where P is a decomposable $\mathbb{P}^2$-bundle over $\mathbb{P}^m$, such that the morphism X/B is the restrition of the $\mathbb{P}^2$-bundle morphism $P \longrightarrow \mathbb{P}^m$ and such that $X \subset P$ is locally given by equations of degree 2 in the $\mathbb{P}^2$-bundle.

Theorem III.F ([BLZ21, Theorem B])**.** *Let $B \subseteq \mathbb{P}^m$ be a smooth projective complex variety, $P \longrightarrow \mathbb{P}^m$ a decomposable $\mathbb{P}^2$-bundle and $X \subset P$ a smooth closed subvariety such that the projection to $\mathbb{P}^m$ induces a conic bundle $\eta : X \longrightarrow B$. Then there exists a homomorphism $\mathrm{Bir}(X) \longrightarrow \bigoplus_{\mathbb{Z}} \mathbb{Z}/2$ whose restriction to $\mathrm{Bir}(X/B)$ is surjective.*

Theorem III.F applies for instance to any smooth cubic X hypersurface of $\mathbb{P}^n$, $n \geqslant 4$, as the blow-up of X in a line carries a decomposable Mori conic bundle structure.

The set $\mathrm{CB}(\mathbb{P}^n)$ of equivalence classes of Mori conic bundles is very large, which allows to construct the following homomorphism.

Theorem III.G ([BLZ21, Proposition 7.15, Theorem E])**.** *For each $n \geqslant 3$, there is an uncountable set J indexing decomposable conic bundles X_i/B_i, where X_i, B_i are rational smooth varieties such that two conic bundles X_i/B_i and X_j/B_j are equivalent if and only if $i = j$. Moreover, there is a surjective homomorphism $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \ast_J \mathbb{Z}/2$ that admits a section. In particular, $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ is a semi-direct product with one factor being a free product.*

In particular, every group generated by an uncoutable set of involutions is a quotient of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$, $n \geqslant 3$.

Every smooth cubic threefold $X \subset \mathbb{P}^4$ is non-rational, and moreover two such cubics are birational if and only if they are equal up to an element of $\mathrm{Aut}(\mathbb{P}^4) = \mathrm{PGL}_5(\mathbb{C})$ [CG72].

Similarly to Theorem III.G, for a general smooth cubic threefold there is a surjective homomorphism $\mathrm{Bir}(X) \longrightarrow \ast_J \mathbb{Z}/2$ with uncountable indexing set J [BLZ21, Proposition 8.9].

III.3 Continuous automorphisms of Cremona groups

In this section, we change our focus from homomorphism from a Cremona group to a product of sums of $\mathbb{Z}/2$ to group homomorphisms from a Cremona group to itself.

Let $\mathbf{k}$ be an arbitrary field. Every field automorphism α of $\mathbf{k}$ naturally induces a group automorphism on both, $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ and $\mathrm{Aut}(\mathbb{P}^n_{\mathbf{k}}) \simeq \mathrm{PGL}_{n+1}(\mathbf{k})$, which we denote by $g \mapsto {}^{\alpha}g$. The group automorphisms of $\mathrm{PGL}_{n+1}(\mathbf{k})$ are well-known: every automorphism of $\mathrm{PGL}_{n+1}(\mathbf{k})$ is the composition of an inner automorphism with an automorphism of the form $g \mapsto {}^{\alpha}g$ or $g \mapsto {}^{\alpha}g^{\vee}$, where α is a field automorphism of $\mathbf{k}$ and $g^{\vee}$ denotes the inverse of the transpose of g [Die71, IV.§1.I–III, p.85–89 and IV.§6, p.98]. Not all automorphisms of $\mathrm{PGL}_{n+1}(\mathbf{k})$ extend to $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$:

Lemma III.3.1 ([Ure18, Corollary A.12])**.** *For any field $\mathbf{k}$ and any field homomorphism α of $\mathbf{k}$, the automorphism of $\mathrm{PGL}_{n+1}(\mathbf{k})$ given by $g \mapsto {}^{\alpha}g^{\vee}$ does not extend to $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$.*

The other automorphisms of $\mathrm{PGL}_{n+1}(\mathbf{k})$ do extend and in fact, any automorphism of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is of this form, up to conjugacy:

Theorem III.3.2 ([Dés06b, Theorem 0.1])**.** *Let φ be an automorphism of the group $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. Then there exists $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ and an automorphism α of the field $\mathbb{C}$ such that $\varphi(g) = f({}^{\alpha}g)f^{-1}$ for all $g \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$.*

Up to date it is unknown whether Theorem III.3.2 also holds for $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$, $n \geqslant 3$.

Note that all automorphism of $\mathrm{PGL}_{n+1}(\mathbf{k})$ are continuous with respect to the Zariski topology.

Cremona groups carry the so-called *Zariski topology*, which was introduced by M. Demazure in [Dem70]. Let X, A irreducible algebraic varieties defined over $\mathbf{k}$. Consider a birational map $f\colon A\times X \dashrightarrow A\times X$ inducing an isomorphism between open dense subsets $U, V \subset A \times X$ such that the restriction of the first projection to U and V is surjective onto A. Every $a \in A(\mathbf{k})$ induces a birational map $f_a\colon X \dashrightarrow X$, $x \mapstochar\dashrightarrow p_2(f(a,x))$, where $p_2\colon A\times X \longrightarrow X$ is the second projection. The map from $A(\mathbf{k})$ to $\mathrm{Bir}(X)$ given by $a \mapsto f_a$ is called a *morphism* (or $\mathbf{k}$*-morphism*) from A to $\mathrm{Bir}(X)$, and is denoted by $A \longrightarrow \mathrm{Bir}(X)$. The *Zariski topology* is now the finest topology such that the preimages of closed subsets by morphisms are closed: A subset $F \subset \mathrm{Bir}(X)$ is closed in the *Zariski topology* if for any algebraic $\mathbf{k}$-variety A and any $\mathbf{k}$-morphism $A \longrightarrow \mathrm{Bir}(X)$ the preimage of F is closed in $A(\mathbf{k})$.

We denote by $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})_{\leqslant d}$ the set of Cremona transformations of degree $\leqslant d$. J. Blanc and J.-P. Furter show in [BF13] that the restriction of the Zariksi topology to $\mathrm{Aut}(\mathbb{P}^n_{\mathbf{k}}) = \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})_1$ is the usual Zariski topology, and provide an equivalent definition of the Zariski topology, in which the sets $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})_{\leqslant d}$ are topological quotients of quasi-projective algebraic varieties given by the coefficients of the birational maps in $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})_{\leqslant d}$. The sets $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})_{\leqslant d}$

are moreover closed in the Zariski topology. It turns out that the Zariski topology on the Cremona groups is the inductive limit topology of the family $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})_{\leqslant d}$, $d \geqslant 1$. While for each $d \geqslant 1$, the set $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})_d$ of Cremona transformations of degree d carries the structure of a variety, the sets $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})_{\leqslant d}$, $d \geqslant 2$, do not carry the structure of a variety. In fact, there is no ind-structure on $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ such that morphisms $A \longrightarrow \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ are morphisms of algebraic varieties [BF13, Theorem 1].

Theorem III.3.2 implies that every group automorphism of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is Zariski continuous. It is unknown whether every group automorphism of $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ is Zariski continuous for $n > 2$ or $\mathbf{k} \neq \mathbb{C}$. The following proposition shows that Zariski-continuous endomorphism of $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ are completely determined by their restriction to $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^n)$.

Let $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ be the group of polynomial automorphisms of the affine space, and by $\mathrm{Aff}(\mathbb{A}^n_{\mathbf{k}}) \subset \mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ we denote the subgroup of affine automorphisms. The *Zariski topology* on $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ is the induced topology of the Zariski topology on $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$.

Proposition III.H ([UZ21, Proposition 3.4, Proposition 3.5])**.** *Let* $\mathbf{k}$ *be an infinite field.*

1. *Let* $\varphi\colon \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}) \longrightarrow \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ *be a group endomorphism which is Zariski continuous. If* $\varphi|_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^n)} = \mathrm{id}_{\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^n)}$*, then* $\varphi = \mathrm{id}_{\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})}$*.*
2. *Let* $\varphi\colon \mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}}) \longrightarrow \mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ *be an endomorphism which is Zariski continuous. If* $\varphi|_{\mathrm{Aff}(\mathbb{A}^n_{\mathbf{k}})} = \mathrm{id}_{\mathrm{Aff}(\mathbb{A}^n_{\mathbf{k}})}$*, then* $\varphi = \mathrm{id}_{\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})}$*.*

Using Proposition III.H, we get a generalisation of Theorem III.3.2, but with the additional restriction that the automorphisms are homeomorphisms with respect to the Zariski topology.

Theorem III.I ([UZ21, Theorem 1.1, Theorem 1.3])**.** *Let* $\mathbf{k}$ *be a field of characteristic* 0 *and let* $\varphi\colon \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}) \longrightarrow \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ *be a group automorphism that is a homeomorphism with respect to the Zariski topology, where* $n \geqslant 2$*. Then there exists a field automorphism* α *of* $\mathbf{k}$ *and an element* $f \in \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ *such that* $\varphi(g) = f\,({}^{\alpha}g)f^{-1}$ *for all* $g \in \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$*.*

The same statement holds for $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ *if* $\mathbf{k}$ *is infinite and perfect.*

Theorem III.I implies that if $\mathbf{k}$ does not have any non-trivial field automorphisms (for instance, if $\mathbf{k} = \mathbb{Q}$ or $\mathbf{k} = \mathbb{R}$), then every automorphisms of $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ that is Zariski continuous is inner.

The main ingredients of the proof of Theorem III.I on $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ is a result of S. Cantat and J. Xie:

Theorem III.3.3 ([CX18, Theorem A, Corollary 8.5])**.** *Let* Γ *of* $\mathrm{SL}_{n+1}(\mathbb{Z})$ *be a finite index subgroup,* $n \geqslant 2$*, and* X *an irreducible complex quasi-projective variety of dimension* n*.*

1. *Given an injective group homomorphism* $\varphi\colon \Gamma \hookrightarrow \mathrm{Bir}(X)$*, there exists a birational map* $f\colon X \dashrightarrow \mathbb{P}^n_{\mathbb{C}}$ *such that* $f\varphi(\Gamma)f^{-1} \subset \mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^n)$*.*
2. *Given an injective group homomorphism* $\varphi\colon \Gamma \hookrightarrow \mathrm{Aut}(X)$*, there exist an isomorphism* $f\colon X \longrightarrow \mathbb{P}^n$ *such that* $f\varphi(\Gamma)f^{-1} \subset \mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^n)$*.*

This result allows us to show that up to conjugation by a suitable birational map, any continuous automorphism φ of $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ maps $\mathrm{Aut}(\mathbb{P}^n_{\mathbf{k}})$ into itself [UZ21, Proposition 4.6]. Theorem III.I then follows from the classification of the automorphisms of $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^n)$, Lemma III.3.1 and Proposition III.H. The proof of Theorem III.I on $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ is less straight forward, but does not use a result as strong as Theorem III.3.3. We show that up to conjugation with an automorphism of $\mathbb{A}^n_{\mathbf{k}}$, any automorphism of $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ that is a homeomorphism with respect to the Zariski topology sends $\mathrm{GL}_n(\mathbf{k})$ to itself. We then show that, again after conjugating with a suitable element of $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$, it sends $\mathrm{Aff}(\mathbb{A}^n_{\mathbf{k}})$ to itself. The claim then follows with Proposition III.H.

In this setting, Theorem III.3.2 and Theorem III.I can be seen as an algebraic analogues of [Fil82], where it is shown that all group automorphisms of diffeomorphism groups are inner.

The group $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ has the additional structure of an *ind-group* (see [FK18] for details). Theorem III.I implies in particular that every ind-group automorphism of $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ is inner, if $\mathbf{k}$ is an infinite perfect field. In the case where $\mathbf{k}$ is of characteristic zero and algebraically closed, this was proven by A. Kanel-Belov, J.-T. Yu, and A. Elishev in [KBYE18]. Again, one can ask, whether all group automorphisms of $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ are Zariski continuous. In dimension 2, the question has a positive answer if $\mathbf{k}$ is uncountable [Dés06a]. A partial generalisation of [Dés06a] to higher dimensions has been obtained in [KS13], [Sta13], [Ure13]. In general, for $n \geqslant 3$ it is an open problem whether all group automorphisms of $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ are inner up to field automorphisms.

A *local field* $\mathbf{k}$ is a field endowed with a locally compact topology that is non-discrete. Any local field is $\mathbb{C}$, $\mathbb{R}$, a finite extension of the p-adic numbers $\mathbb{Q}_p$ or the field of formal Laurent series over a finite field [Mil, Remark 7.49]. In [BF13, §5], J. Blanc and F. Furter define a natural refinement of the Zariski topology on Cremona groups over a local field $\mathbf{k}$, namely the *Euclidean topology*, in which the $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})_{\leqslant d}$ are topological quotients of manifolds given by the coefficients of its the birational maps in $\mathrm{Bir}(\mathbb{P}^n)_{\leqslant d}$. The induced topology on $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^n)$ is the usual Euclidean topology, and it makes $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ a Hausdorff topological group which is not metrisable. Moreover, any compact subset of $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ is of bounded degree [BF13, §5].

Proposition III.H(1) also holds in the Euclidean topology over any local field, and Theorem III.I on $\mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ holds for $\mathbf{k} = \mathbb{C}$ and $\mathbf{k} = \mathbb{R}$ [UZ21, Theorem 1.2, Proposition 3.4]. Its proof requires $\mathbf{k}$ to be archimedean and we did not find a proof over non-archimedean local fields.

IV Structures of Cremona groups

There are many beautiful structure theorems for the plane Cremona group over an algebraically closed field. For a selection of results, see for instance [UZ19]. Notably, the following is known:

Theorem IV.0.1 ([CL13, Corollary A.2])**.** *If $\mathbf{k}$ is algebraically closed, then $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is not isomorphic to a non-trivial free product of groups amalgamated along their common intersection*

The plane Cremona group comes close to being an amalgam. Let $\mathcal{J}_* \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ be the subgroup of elements preserving the pencil of lines through a point.

Theorem IV.0.2 ([Bla12]). *$\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is the amalgamated product of $\mathcal{J}_*$ and $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ modulo the relation $\sigma \circ \tau = \tau \circ \sigma$, where σ is the standard quadratic transformation and $\tau \in \mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ the transposition $\tau\colon [x:y:z] \mapsto [y:x:z]$.*

The plane Cremona group is a generalised amalgamated product.

Theorem IV.0.3 ([Wri92, Theorem 3.13]). *If $\mathbf{k}$ is algebraically closed, then $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is the free product of $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$, $\mathrm{Aut}(\mathbb{P}^1_{\mathbf{k}} \times \mathbb{P}^1_{\mathbf{k}})$ and $\mathcal{J}_*$ amalgamated along their pairwise intersections in $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$.*

In this section we will explain that the Cremona groups in dimension 2 over non-closed fields and in dimension $\geqslant 3$ are indeed non-trivial amalgams of two groups. The structure theorems appearing in this section are related to the non-trivial homomorphism of groups in §III.1 and §III.2.

IV.1 Plane Cremona groups over a non-closed perfect field

A field $\mathbf{k}$ that is not algebraically closed but whose algebraic closure $\overline{\mathbf{k}}$ is a finite extension of $\mathbf{k}$, is called *real closed field*. It satisfies $[\overline{\mathbf{k}}:\mathbf{k}] = 2$ and is of characteristic zero by [AS27, Satz 4]. Let $\mathcal{S}$ be the del Pezzo surface obtained by blowing up two points of degree 2 in $\mathbb{P}^2$ and contracting the strict transform of the line through one of them. It carries a Mori conic bundle structure $\mathcal{S}/\mathbb{P}^1$ given by the conics through the two points.

Consider a Sarkisov link $\mathcal{S} \dashrightarrow \mathcal{S}$ of type II over $\mathbb{P}^1$ whose base-locus is a point of degree 2 not contained in the exceptional double section of $\mathcal{S}/\mathbb{P}^1$. It is conjuguate to a birational map of $\mathbb{P}^2$ of degree 5 not defined at three points of degree 2 that contracts six $\overline{\mathbf{k}}$-conics, and which is called *standard quintic transformation*.

We have the following generating set of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ of bounded degree.

Theorem IV.1.1 ([BM14, Theorem 1.2]). *Let $\mathbf{k}$ be a real closed field. The group $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is generated by $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$, the two quadratic involutions $\sigma\colon [x:y:z] \dashrightarrow [yz:xz:xy]$ and $\tau\colon [x:y:z] \dashrightarrow [xz:yz:x^2+y^2]$, and the set of standard quintic transformations.*

Theorem IV.1.1 is proven by J. Blanc and F. Mangolte in [BM14, Theorem 1.2] for $\mathbf{k} = \mathbb{R}$. Its proof relies on the classification of rational Mori fibre spaces and Sarkisov links between them. The classification is the same over any real closed field by Proposition V.B and Lemma III.1.6, because a real closed field has a unique non-trivial finite field extension, which is quadratic. The group $\mathrm{Bir}(\mathcal{S}/\mathbb{P}^1)$ is conjugate to the group $\mathcal{J}_\circ \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ of elements preserving the pencil of conics through two fixed points of degree 2 in $\mathbb{P}^2$ whose geometric components are in general position. It contains a conjugate of τ and a conjugate of each standard quintic transformation. The group $\mathrm{Bir}(\mathbb{F}_1/\mathbb{P}^1)$ is conjugate to the subgroup $\mathcal{J}_* \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ of elements preserving the pencil of lines through a fixed rational point. It contains a conjugate of τ and of σ. As consequence of Theorem IV.1.1, $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is generated by $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$, $\mathcal{J}_*$ and $\mathcal{J}_\circ$. This is also the generating set of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ constructed in [Isk91], if considered over a real closed field.

Using the list of elementary relations of Sarkisov links in $\mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}})$ over a real closed field $\mathbf{k}$, see Example II.3.2, one shows that $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is a free product of two groups amalgamated along their common intersection - contrary to the situation over $\mathbb{C}$.

Theorem IV.A ([Zim18b, Theorem 1.1])**.** *Let $\mathbf{k}$ be a real closed field. Then $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is the free product of the groups $\mathcal{G}_* := \langle \mathcal{J}_*, \mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})\rangle$ and $\mathcal{G}_\circ := \langle \mathcal{J}_\circ, \mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})\rangle$ amalgamated along their intersection $\mathcal{G}_* \cap \mathcal{G}_\circ$, which is equal to the group $\langle \mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}}), \tau\rangle$. Moreover, both $\mathcal{G}_*$ and $\mathcal{G}_\circ$ have uncountable index.*

Let $\mathbf{k}$ be an arbitrary perfect field and let $\mathcal{B} \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ a set of representatives of Bertini involutions up to conjugacy with $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$. For each $b \in \mathcal{B}$, define $G_b := \langle b, \mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})\rangle$. We denote by $G_e \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ the subgroup generated by $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ and the conjugates of Sarkisov links between rational Mori fibre spaces of dimension 2 that are not conjugate to Bertini links from $\mathbb{P}^2$ to $\mathbb{P}^2$. This is well defined, as Bertini links between $\mathbb{P}^2$ are not conjugate to any other Sarkisov links, because they do not appear in any non-trivial elementary relation, see Example II.3.3.

Theorem IV.B ([LZ20, Theorem A, Corollary B, Theorem C])**.** *Let $\mathbf{k}$ be a perfect field with a Galois extension of degree 8, and consider the groups G_i, $i \in \mathcal{B} \cup \{e\}$, as defined above.*

1. *Then $G_i \cap G_j = \mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ for all $i \neq j$ and the Cremona group is the amalgamated product of the G_i along their common intersection $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \simeq \mathop{\ast}_{\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})} G_i$, and $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ acts faithfully on the corresponding Basse-Serre tree.*
2. *Let $\mathcal{G}_{\mathcal{B}} = \langle \mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}}), \mathcal{B}\rangle$. Then $\mathcal{G}_{\mathcal{B}} \cap G_e = \mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})$ and the Cremona group is isomorphic to the amalgamated product $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \simeq \mathcal{G}_{\mathcal{B}} *_{\mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}})} G_e$, and $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ acts faithfully on its Bass-Serre tree.*
3. *For each $b \in \mathcal{B}$, we have $G_b \simeq \mathrm{Aut}(\mathbb{P}^2_{\mathbf{k}}) * \mathbb{Z}/2$, and we can write the Cremona group as free product $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}}) \simeq G_e * (\mathop{\ast}_{\mathcal{B}} \mathbb{Z}/2)$.*

We obtain Theorem III.A as corollary of Theorem IV.B.

For a perfect field $\mathbf{k}$ and a Mori fibre space X over $\mathbf{k}$ of $\dim X = 2$ we denote by ρ the group homomorphism

$$\rho\colon\ \mathrm{Bir}(X) \longrightarrow \mathop{\ast}_{C \in \mathrm{CB}(X)} \bigoplus_{M(C)} \mathbb{Z}/2$$

from Theorem III.1.4. For each $C \in \mathrm{CB}(X)$ we fix a choice of representative X_C/B_C, and we define $G_C := \rho^{-1}(\rho(\mathrm{Bir}(X_C/B_C))) \subseteq \mathrm{Bir}(X)$. The following result is a consequence of Theorem III.1.4 and Theorem III.1.7.

Theorem IV.1.2. *Let $\mathbf{k}$ be a perfect field with $[\overline{\mathbf{k}} : \mathbf{k}] > 2$ and let X/B be a smooth Mori conic bundle over $\mathbf{k}$ of dimension $\dim X = 2$. Then, the following hold:*

1. *For all $C \neq C'$ in $\mathrm{CB}(X)$, the group $A = G_C \cap G_{C'}$ contains $\ker \rho$ and does not depend on the choice of C and C'.*

2. *The group* $\mathrm{Bir}(X) = \ast_A G_C$ *is the free product of the groups* G_C*,* $C \in \mathrm{CB}(X)$*, amalgamated over their common intersection* A*.*
3. $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ *is the free product of* $G_{\mathbb{F}_1}$*, the* $G_{\mathcal{S}}$ *and the* $G_{\mathcal{X}}$*, amalgamated along their common intersection* A*, and the product is non-trivial.*

Proof. (1)&(2) are analogous to the proof of Theorem IV.C, see [BLZ21, Theorem 8.6], by Theorem III.1.4. (3) By Lemma III.1.6, $\mathrm{CB}(\mathbb{P}^2)$ consists of the class of $\mathbb{F}_1/\mathbb{P}^1$, and the classes of the form $\mathcal{S}/\mathbb{P}^1$ and $\mathcal{X}/\mathbb{P}^1$. We have $A \subsetneq G_{\mathbb{F}_1}, G_{\mathcal{S}}, G_{\mathcal{X}} \subsetneq \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}})$ by Theorem III.1.7, so the amalgamated product is non-trivial. □

Note that there may be countably many classes of the form $\mathcal{S}/\mathbb{P}^1$ and $\mathcal{X}/\mathbb{P}^1$, so the amalgamated product may have countably many factors. It is not clear that $A = \ker\rho$ in Theorem IV.1.2, because some elements of $\bigoplus_{MC(C)} \mathbb{Z}/2$ may be in the image of $\mathrm{Bir}(X)$ but not in the image of $\mathrm{Bir}(X/B)$.

IV.2 Cremona groups in higher dimension

Contrary to $n = 2$ and $\mathbf{k} = \mathbb{C}$, and analogously to $n = 2$ and $\mathbf{k}$ a non-closed perfect field, the Cremona group in higher dimension is a free product of groups amalgamated over their common intersection.

For a terminal variety X of dimension $n \geqslant 3$, an element C of $\mathrm{CB}(X)$ is called *decomposable* if it is the class of a decomposable conic bundle (Definition III.2.3). We denote by ρ the group homomorphism

$$\rho\colon \mathrm{Bir}(X) \longrightarrow \mathop{\ast}_{C\in\mathrm{CB}(X)} \bigoplus_{M(C)} \mathbb{Z}/2$$

from Theorem III.E. For each $C \in \mathrm{CB}(X)$ we fix a choice of representative X_C/B_C, and we denote $G_C = \rho^{-1}(\rho(\mathrm{Bir}(X_C/B_C))) \subseteq \mathrm{Bir}(X)$.

Theorem IV.C ([BLZ21, Proposition 8.6])**.** *For each* $n \geqslant 3$*, and let* X/B *be a conic bundle, where* X *is a terminal variety of dimension* n*. Then, the following hold:*

1. *For all* $C \neq C'$ *in* $\mathrm{CB}(X)$*, the group* $A = G_C \cap G_{C'}$ *contains* $\ker\rho$ *and does not depend on the choice of* C *and* C'*;*
2. *The group* $\mathrm{Bir}(X) = \ast_A G_C$ *is the free product of the groups* G_C*,* $C \in \mathrm{CB}(X)$*, amalgamated over their common intersection* A*.*
3. *For each decomposable* $C \in \mathrm{CB}(X)$ *we have* $A \subsetneq G_C$*. Moreover, the free product of (2) is non-trivial (i.e.* $A \subsetneq G_C \subsetneq \mathrm{Bir}(X)$ *for each* C*) as soon as* $\mathrm{CB}(X)$ *contains two distinct decomposable elements.*

Again, it is not clear that $A = \ker\rho$. Theorem IV.C applies in particular when X is rational, as $\mathrm{CB}(X)$ then contains uncountably many decomposable elements by Theorem III.G.

A classical result, due to H. Hudson and I. Pan [Hud27, Pan99], says that for $n \geqslant 3$ and $\mathbf{k} = \mathbb{C}$, the group $\mathrm{Bir}(\mathbb{P}^n)$ is not generated by any set of elements of $\mathrm{Bir}(\mathbb{P}^n)$ of bounded degree. Indeed, for each irreducible hypersurface $\Gamma \subset \mathbb{P}^{n-1}$, pick a homogeneous irreducible polynomial $p \in \mathbf{k}[x_2, \ldots, x_n]$ that defines Γ. The birational

$$f_p\colon (x_1, \ldots, x_n) \dashrightarrow (x_1 p(x_2, \ldots, x_n), x_2, \ldots, x_n)$$

contracts a hypersurface birational to $\mathbb{P}^1 \times \Gamma$ onto $\{0\} \times \Gamma$. So, to generate the Cremona group we need at least one generator for each birational class of irreducible hypersurfaces of $\mathbb{P}^{n-1}$. The map f_p preserves a family of hyperplanes through $[0 : 1 : \cdots : 0]$. The set of elements of $\mathrm{Bir}(\mathbb{P}^n)$ preserving a family through a given point forms a group isomorphic to $\mathrm{PGL}_2(\mathbb{C}(x_2, \ldots, x_n)) \rtimes \mathrm{Bir}(\mathbb{P}^{n-1})$, and its elements are called *Jonquières*. It is a natural question whether the group $\mathrm{Bir}(\mathbb{P}^n)$ is generated by $\mathrm{Aut}(\mathbb{P}^n)$ and by Jonquières elements (asked in [PS15], for instance). The answer to the question is negative, just like in dimension $n = 2$ over a non-closed perfect field (see Theorem III.1.2 and Theorem III.1.7).

Theorem IV.D ([BLZ21, Theorem C])**.** *Let $n \geqslant 3$. Let S be a set of elements in the Cremona group $\mathrm{Bir}(\mathbb{P}^n)$ that has cardinality strictly smaller than $|\mathbb{C}|$, and let $G \subseteq \mathrm{Bir}(\mathbb{P}^n)$ be the subgroup generated by $\mathrm{Aut}(\mathbb{P}^n)$, all Jonquières elements and S. Then G is contained in the kernel of a surjective group homomorphism $\mathrm{Bir}(\mathbb{P}^n) \longrightarrow \mathbb{Z}/2$. In particular, G is a proper subgroup of $\mathrm{Bir}(\mathbb{P}^n)$, and the same is true for the normal subgroup generated by G.*

Using Theorem III.2.1, J. Blanc and E. Yasinksy prove a stronger statement for $n = 3$. They show that the subgroup $G \subset \mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$ generated by all elements preserving a fibration $\mathbb{P}^3_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ whose general fibres are rational, is in fact a strict subgroup of $\mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$ [BY20, Theorem B]. As consequence, they show that the subgroup of $\mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$ generated by all connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$ is a strict normal subgroup of $\mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$ [BY20, Theorem C].

V Algebraic subgroups of the plane Cremona group

The classification of algebraic groups acting on $\mathbb{P}^2_{\mathbf{k}}$ up to conjugacy is open over many fields, because classifying finite groups acting birationally on $\mathbb{P}^2_{\mathbf{k}}$ is very hard. There are several classifiction results for finite and abelian groups [BB00, BB04, Bla07, DI09b, Bla09a], and the classification had been completed over algebraically closed fields in the works of J. Blanc, and I. Dolgachev and V.A. Iskovskikh. Over non-closed fields, only partial classifications exist, and they can be found in [DI09a, Rob16, Yas16, Yas19].

The classification of infinite algebraic groups acting birationally on $\mathbb{P}^2_{\mathbb{C}}$ up to conjugacy and up to inclusion has been achieved in [Bla09b], and $\mathbf{k} = \mathbb{R}$ it can be found in [RZ18]. In higher dimension, the *connected* algebraic groups acting birationally on $\mathbb{P}^3_{\mathbb{C}}$ have been classified up to conjugation and inclusion by H. Umemura in [Ume80, Ume82a, Ume82b, Ume85]. The classification has been recovered and extended to closed fields of characteristic zero in [BFT17, BFT19] by J. Blanc, A. Fanelli and R. Terpereau in a different approach using the MMP. An attack in dimension 4 has been started in [BF20] by J. Blanc and E. Floris.

We present here the classification of the *infinite* algebraic groups acting birationally on a rational smooth projective surface over a perfect field $\mathbf{k}$ *up to conjugation and up to inclusion*.

It will turns out that any algebraic group acting birationally on $\mathbb{P}^2_{\mathbf{k}}$ acts regularly on a rational smooth projective surface X that is either a del Pezzo surface of degree $\geqslant 6$ or a special conic fibration, see Proposition V.A and Theorem V.H. The automorphism groups of these surfaces affine algebraic groups, and this reduces the classification problem to the task of classifying these surfaces X, and classify their automorphism groups up to conjugation by birational maps. An equivariant version of Theorem II.1.10(1), see [Zim18b, §7.1], tells us that it is enough to study $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant Sarkisov links starting from X and list all possible options to go equivariantly from one such surface X to another. It turns out that equivariant Sarkisov links only exist for very special X, which we specify in Theorem V.I.

Throughout this chapter, the base field $\mathbf{k}$ is a perfect field.

V.1 Birational group actions

We say that an algebraic group G *acts birationally* on a variety X if there are open dense subsets $U, V \subset G \times X$ and a birational map

$$G \times X \dashrightarrow G \times X, \quad (g, x) \mapstochar\dashrightarrow (g, \rho(g, x))$$

restricting to a isomorphism $U \xrightarrow{\simeq} V$ and the projection of U and V to the first factor is surjective onto G, and $\rho(e, \cdot) = \mathrm{id}_X$ and $\rho(gh, x) = \rho(g, \rho(h, x))$ for any $g, h \in G$ and $x \in X$ such that $\rho(h, x), \rho(gh, x)$ and $\rho(g, \rho(h, x))$ are well defined. This is equivalent to saying that there is a *morphism* $G \longrightarrow \mathrm{Bir}(X)$ (see §III.3), such that the induced map $G(\mathbf{k}) \longrightarrow \mathrm{Bir}(X)$, $g \mapsto \rho(g, \cdot)$, is a homomorphism of groups. If the birational map $G \times X \dashrightarrow G \times X$ above is an isomorphism, we say that X is a G-variety.

For a projective surface X, we denote by $\mathrm{Aut}_{\mathbf{k}}(X)$ the group of $\mathbf{k}$-automorphisms of X, which is the group of $\mathbf{k}$-rational points of a group scheme $\mathrm{Aut}(X)$ that is locally of finite type over $\mathbf{k}$ [Bri17b, Theorem 7.1.1]. If X is a G-surface, then $G(\mathbf{k}) \ni g \mapsto \rho(g, \cdot)$ induces a homomorphism of groups $G(\mathbf{k}) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X)$.

Any algebraic group acting birationally on $\mathbb{P}^2$ is an affine algebraic group [BF13, §2.6], and so the next proposition allows us to restrict to studying rational G-surfaces. It was proven separately by A. Weil and M. Rosenlicht in [Wei55, Ros56], but neither of them needed the new model to be smooth nor projective. The proof of the first part of the following statement was communicated to us by M. Brion, and it holds over any (not necessarily perfect) field.

Proposition V.1.1. *Let X be a smooth projective surface and G an affine algebraic group acting birationally on X. Then there exists a G-surface Y and a G-equivariant birational map $X \dashrightarrow Y$. Furthermore, $G(\mathbf{k})$ has finite action on* $\mathrm{NS}(Y)$.

Proof. By [Wei55, Ros56], there exists a normal G-surface Y' and a G-equivariant morphism $X \dashrightarrow Y'$. The set Y'' of smooth points of Y' is G-stable, it is contained in a complete surface, which can be desingularised [Lip78], so Y' is quasi-projective. By [Bri17a, Corollary 2.14], Y' has a G-equivariant completion Y''. We now desingularise Y'' G-equivariantly and obtain the smooth projective surface Y [Zar39, Lip69] (the sequence of blow-ups and normalisations over $\mathbf{k}$ can be done G-equivariantly).

The second claim is classical and for instance shown in [RZ18, Lemma 2.10] over any perfect field. □

We can now replace $\mathbb{P}^2$ with a suitable rational smooth projective G-surface X. Moreover, after quotienting out the schematic kernel of the G-action, we can assume that the action is faithful.

We can view the G-action on X as $(G_{\overline{\mathbf{k}}} \times \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))$-action on $X_{\overline{\mathbf{k}}}$. It is a finite action, so we can start the *G-equivariant MMP* from X over $\mathbf{k}$. This means that we successively contract $(G(\overline{\mathbf{k}}) \times \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))$-orbits of disjoint (-1)-curves. At each contraction, the Picard rank drop, and the process ends on a rational smooth projective surface Y. It satisfies one of the following properties and is called *G-Mori fibre space*,

- $-K_Y$ is ample and $N^1(X_{\overline{\mathbf{k}}})^{G_{\overline{\mathbf{k}}} \times \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})} = 1$,
- there is a surjective morphism $\pi\colon Y \longrightarrow \mathbb{P}^1$ with connected fibres such that $-K_Y$ is π-ample and $\mathrm{rkNS}(X_{\overline{\mathbf{k}}})^{G_{\overline{\mathbf{k}}} \times \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})} = 1$.

We can now run the *$G(\mathbf{k})$-equivariant MMP* from Y over $\mathbf{k}$, that is, we successively contract all $(G(\mathbf{k}) \times \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}))$-orbits of disjoint (-1)-curves. This process ends on a rational smooth projective surface Z, which satisfies one of the properties above with $G_{\overline{\mathbf{k}}}$ replaced by $G(\mathbf{k})$, and is called *$G(\mathbf{k})$-Mori fibre space*.

If G is connected, Blanchard's Lemma [BSU13, Proposition 4.2.1] implies that for any extremal contraction $X \longrightarrow Y$ there exists a unique G-action on Y making the contraction G-equivariant. In particular, for G connected, the G-MMP is the usual MMP and a G-Mori fibre space is a Mori fibre space. If G is not connected, there are G-Mori fibre spaces that are not $G(\mathbf{k})$-Mori fibre spaces, nor Mori fibre spaces. An example is a del Pezzo surface X of degree 6 from Proposition V.E, which is a $\mathrm{Aut}(X)$-Mori fibre space by Proposition V.D(1) but not an $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space nor a Mori fibre space.

If $X \longrightarrow \mathbb{P}^1$ is a surjective morphism with connected fibres, we denote by $\mathrm{Aut}(X, \mathbb{P}^1) \subseteq \mathrm{Aut}(X)$ the subgroup preserving this fibration.

Recall from Definition III.1.5 that we denote by $\mathcal{S}^{L,L}$ a del Pezzo surface of degree 6 obtained by blowing up $\mathbb{P}^2$ in two points of degree 2 with splitting field $L/\mathbf{k}$ and $L'/\mathbf{k}$ respectively, and then contracting the strict transform of a line through one of them. The conics through the two points induce a conic bundle structure $\mathcal{S}^{L,L'}/\mathbb{P}^1$.

Theorem V.1.1 and then applying the G-MMP and the classification of rational minimal conic bundles from Lemma III.1.6 implies the following proposition.

Proposition V.A ([SZ21, Proposition 2.13])**.** *Let G an infinite algebraic group acting birationally on $\mathbb{P}^2$. Then there exists a G-equivariant birational map $\mathbb{P}^2 \dashrightarrow X$ to a G-Mori fibre space X/B that is one of the following:*

1. *B is a point and X a del Pezzo surface of degree $K_X^2 \in \{6, 8, 9\}$ and* $\mathrm{rkNS}(X_{\overline{\mathbf{k}}})^{\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}) \times G_{\overline{\mathbf{k}}}} = 1$.

2. *$B = \mathbb{P}^1$ and there exists a birational morphism of conic fibrations $X \longrightarrow \mathbb{F}_n$ for some $n \geqslant 0$, or $X \longrightarrow \mathcal{S}^{L,L'}$ for some quadratic extensions $L/\mathbf{k}, L'/\mathbf{k}$, and $\mathrm{rkNS}(X_{\overline{\mathbf{k}}})^{\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})\times G_{\overline{\mathbf{k}}}} = 2$ and $G \subseteq \mathrm{Aut}(X, \mathbb{P}^1)$.*

If X is a del Pezzo surface, then $\mathrm{Aut}(X)$ is an affine algebraic group, and if $\pi\colon X \longrightarrow \mathbb{P}^1$ is a geometrically rational conic fibration, then $\mathrm{Aut}(X, \mathbb{P}^1)$ is an affine algebraic group, see for instance [SZ21, Lemma 2.14]. So, to classify up to conjugation and inclusion the infinite algebraic groups acting birationally and faithfully on $\mathbb{P}^2$ it suffices to classify the surfaces in Proposition V.A, to describe their automorphism groups, and then to classify these groups up to conjugation by equivariant birational maps.

V.2 The relatively minimal surfaces

We present the classification of the rational del Pezzo surfaces of degree 6 and 8, and the rational conic fibrations $X/\mathbb{P}^1$ that are $\mathrm{Aut}(X, \mathbb{P}^1)$-Mori fibre spaces.

V.2.1 Del Pezzo surfaces

Suppose that $\mathbf{k}$ has a quadratic extension $L/\mathbf{k}$. We denote by $\mathcal{Q}^L$ the $\mathbf{k}$-form $\mathbb{P}^1_L \times \mathbb{P}^1_L$ given by $([u_0 : u_1], [v_0 : v_1]) \mapsto ([v_0^g : v_1^g], [u_0^g : u_1^g])$, where g is the generator of $\mathrm{Gal}(L/\mathbf{k})$. The point $([1 : 1], [1 : 1])$ is a $\mathbf{k}$-rational point of $\mathcal{Q}^L$, so $\mathcal{Q}^L$ is rational over $\mathbf{k}$. The second part of the following lemma is classical, see for instance [Poo17, Proposition 9.4.12].

Proposition V.B ([SZ21, §3])**.**

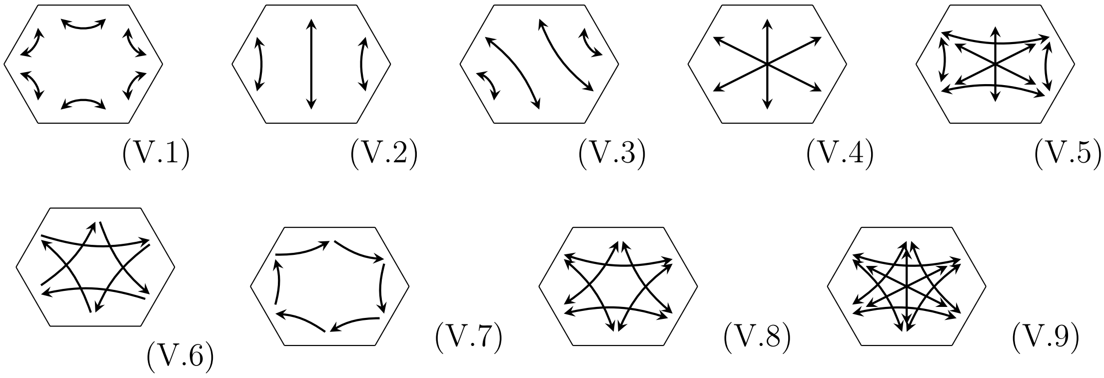
1. *$\mathcal{Q}^L$ is $\mathbf{k}$-isomorphic to the quadric in $\mathbb{P}^3_{wxyz}$ given by $wz = x^2 + axy + \tilde{a}y^2$, where $t^2 + at + \tilde{a} \in \mathbf{k}[t]$ is the minimal polynomial of an element of $L\backslash\mathbf{k}$.*
2. *If $L/\mathbf{k}$ and $L'/\mathbf{k}$ are quadratic extensions, then $\mathcal{Q}^L \simeq \mathcal{Q}^{L'}$ if and only if L and L' are $\mathbf{k}$-isomorphic.*
3. *A rational del Pezzo surface of degree 8 is isomorphic to $\mathbb{F}_0, \mathbb{F}_1$ or some $\mathcal{Q}^L$.*
4. *The group $\mathrm{Aut}(\mathcal{Q}^L)$ is isomorphic the $\mathbf{k}$-form on $\mathrm{Aut}(\mathbb{P}^1_L \times \mathbb{P}^1_L) \simeq \mathrm{Aut}(\mathbb{P}^1_L)^2 \rtimes \langle (u, v) \overset{\tau}{\mapsto} (v, u)\rangle$ given by the $\mathrm{Gal}(L/\mathbf{k})$-action $(A, B, \tau)^g = (B^g, A^g, \tau)$.*

The next few results are the classification rational of del Pezzo surfaces X of degree 6 and a description of $\mathrm{Aut}(X)$ and $\mathrm{Aut}_{\mathbf{k}}(X)$. Recall that $X_{\overline{\mathbf{k}}}$ contains precisely six (-1)-curves, which can be represented by a hexagon, and on which $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ acts by symmetries.

Proposition V.2.1 ([SZ21, §4], [RZ18, §3])**.** *If X is a rational del Pezzo surface of degree 6, then the action of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ on the (-1)-curves of $X_{\overline{\mathbf{k}}}$ is one of the actions in Figure V.1. Moreover, each of these actions are realised on some del Pezzo surface of degree 6 over some perfect field.*

If X is a rational del Pezzo surface of degree 6, the action of $\mathrm{Aut}_{\overline{\mathbf{k}}}(X)$ on $\mathrm{Pic}(X_{\overline{\mathbf{k}}})$ induces the split exact sequence

$$1 \to (\overline{\mathbf{k}}^*)^2 \longrightarrow \mathrm{Aut}_{\overline{\mathbf{k}}}(X) \longrightarrow \mathrm{Sym}_3 \times \mathbb{Z}/2 \to 1,$$



(V.1) (V.2) (V.3) (V.4) (V.5)

(V.6) (V.7) (V.8) (V.9)

Figure V.1: The $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-actions on the hexagon of a del Pezzo surface of degree 6.

and $\mathrm{NS}(X_{\overline{\mathbf{k}}})^{\mathrm{Aut}_{\overline{\mathbf{k}}}(X)} = 1$. We now establish the restriction of the sequence to $\mathrm{Aut}_{\mathbf{k}}(X)$ for each of the cases in Figure V.1. The following propositions are a summary of [SZ21, §4], and they are ordered according to the invariant Picard rank, and $\mathrm{Aut}_{\mathbf{k}}(Y, p_1, \ldots, p_r) \subset \mathrm{Aut}_{\mathbf{k}}(Y)$ denotes the subgroup of elements fixing $p_1, \ldots, p_r \in Y_{\overline{\mathbf{k}}}$, and $\mathrm{Aut}_{\mathbf{k}}(Y, \{p_1, \ldots, p_r\})$ is the subgroup of elements preserving the set $\{p_1, \ldots, p_r\}$.

Proposition V.C ([Zim18b, §4]). *Let X be a rational del Pezzo surface of degree 6. If* $\mathrm{rkNS}(X) = 1$*, there is a quadratic extension $L/\mathbf{k}$ and a birational morphism $\pi \colon X_L \longrightarrow \mathbb{P}^2_L$ blowing up a point $p = \{p_1, p_2, p_3\}$ of degree 3 with splitting field $F/\mathbf{k}$, and one of the following cases holds:*

1. *X is as in Figure (V.7),* $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathbb{Z}/3$ *and the action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X_{\overline{\mathbf{k}}})$ *induces the split exact sequence*

$$1 \to \mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}} \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathbb{Z}/6 \to 1$$

2. *X is as in Figure (V.9),* $\mathrm{Gal}(F/\mathbf{k}) \simeq \mathrm{Sym}_3$ *and the action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X_{\overline{\mathbf{k}}})$ *induces the split exact sequence where $\mathbb{Z}/2$ is generated by a rotation*

$$1 \to \mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)^{\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}} \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathbb{Z}/2 \to 1,$$

and $\pi\,\mathrm{Gal}(L/\mathbf{k})\pi^{-1}$ acts by conjugation on $\mathrm{Aut}_L(\mathbb{P}^2, p_1, p_2, p_3)$. Moreover, X is a $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space.

The first of the following statements is classical.

Proposition V.D ([Zim18b]). *Let X be a rational del Pezzo surface of degree 6. If* $\mathrm{rkNS}(X) \geqslant 2$ *and* $\mathrm{rkNS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 1$*, then X is a $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space, and X is one of the following:*

1. *X is as in Figure (V.1), it is the blow-up of $\mathbb{P}^2$ in three rational points, and the action of* $\mathrm{Aut}_{\mathbf{k}}(X)$ *on* $\mathrm{NS}(X)$ *induces the split exact sequence*

$$1 \to (\mathbf{k}^*)^2 \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathrm{Sym}_3 \times \mathbb{Z}/2 \to 1.$$

2. *X is as in Figure (V.4), it is the blow-up of $\mathbb{F}_0$ in a point $p = \{(p_1, p_1), (p_2, p_2)\}$ of degree 2 and the action of $\mathrm{Aut}_{\mathbf{k}}(X)$ on $\mathrm{NS}(X)$ induces the exact sequence,*

$$1 \to \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, p_1, p_2)^2 \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathrm{Sym}_3 \times \mathbb{Z}/2 \to 1$$

which is split if $\mathrm{char}(\mathbf{k}) \neq 2$.

3. *X is as in Figure (V.6), it is the blow-up of a point $p = \{p_1, p_2, p_3\}$ of degree 3 in $\mathbb{P}^2$ with splitting field L such that $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathbb{Z}/3$. The action of $\mathrm{Aut}_{\mathbf{k}}(X)$ on $\mathrm{NS}(X)$ induces the split exact sequence*

$$1 \to \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathbb{Z}/6 \to 1$$

4. *X is as in Figure (V.8), it is the blow-up of a point $p = \{p_1, p_2, p_3\}$ of degree 3 in $\mathbb{P}^2$ with splitting field L such that $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathrm{Sym}_3$. The action of $\mathrm{Aut}_{\mathbf{k}}(X)$ on $\mathrm{NS}(X)$ induces the split exact sequence, where $\mathbb{Z}/2$ is generated by a rotation*

$$1 \to \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2, p_1, p_2, p_3) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathbb{Z}/2 \to 1$$

Proposition V.E ([Zim18b])**.** *Let X be a rational del Pezzo surface of degree 6. If $\mathrm{rkNS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} \geqslant 2$, then X is as in Figure (V.2),(V.3),(V.5). Moreover, there is a birational morphism $\nu\colon X \longrightarrow \mathcal{Q}^L$ contracting two rational curves onto two rational points p_1, p_2 or an irreducible curve onto a point $\{p_1, p_2\}$ of degree 2. The action of $\mathrm{Aut}_{\mathbf{k}}(X)$ on $\mathrm{NS}(X)$ induces the split exact sequence*

$$1 \to T^{L,L'}(\mathbf{k}) \longrightarrow \mathrm{Aut}_{\mathbf{k}}(X) \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1,$$

where $\eta T^{L,L'}\nu^{-1} \subset \mathrm{Aut}(\mathcal{Q}^L, p_1, p_2)$ is the subgroup preserving the rulings of $\mathcal{Q}^L$ and $L'/\mathbf{k}$ is the splitting field of the set $\{p_1, p_2\}$, and $\nu\,\mathrm{Aut}_{\mathbf{k}}(X)\nu^{-1} = \mathrm{Aut}_{\mathbf{k}}(\mathcal{Q}, \{p_1, p_2\})$ and $\mathrm{rkNS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 2$.

If p_1, p_2 are rational, then $L' = \mathbf{k}$ and $T^{L,\mathbf{k}}$ is the split torus of dimension 2. If $\{p_1, p_2\}$ is a point of degree 2 whose splitting field L' is $\mathbf{k}$-isomorphic to L, then $T^{L,L'}$ is a 2-dimensional split torus as well. This is not the case if L and L' are not $\mathbf{k}$-isomorphic, as then $T^{L,L'}(\mathbf{k})$ contains elements that are not diagonisable over $\mathbf{k}$.

V.2.2 Conic fibrations

Recall that the minimal conic fibration $\mathcal{S}^{L,L'}/\mathbb{P}^1$ can be obtained by blowing up the quadric surface $\mathcal{Q}^L$ in a point of degree 2 with splitting field L' whose components are not in the same ruling of $\mathcal{Q}^L_L$. It is isomorphic to a del Pezzo surface of degree 6 from (V.3) or (V.5). When viewed $\mathcal{Q}^L$ as quadratic surface in $\mathbb{P}^3$, then

$$\mathcal{S}^{L,L'} \simeq \{[w : x : y : z], [u : v] \in \mathcal{Q}^L \times \mathbb{P}^1 \mid u(w + bx + \tilde{b}z) = vy\}$$

where $t^2 + bt + \tilde{b} \in \mathbf{k}[t]$ is the minimal polynomial of an element of $L' \backslash \mathbf{k}$ [SZ21, §3]. The fibration $\mathcal{S}^{L,L'}/\mathbb{P}^1$ is then given by the second projection.

For a conic bundle $X/\mathbb{P}^1$, we have a homomorphism $\mathrm{Aut}(X,\mathbb{P}^1) \longrightarrow \mathrm{Aut}(\mathbb{P}^1)$, whose kernel we denote by $\mathrm{Aut}(X/\mathbb{P}^1)$, and by $\mathrm{Aut}_{\mathbf{k}}(X,\mathbb{P}^1)$ its set of $\mathbf{k}$-points.

Next we describe rational Mori conic bundles $X \longrightarrow \mathbb{P}^1$ such that $\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\mathbb{P}^1)$ acts non-trivially on the set of components of the singular fibres. This is a generalisation of [Bla09b, Lemme 4.3.5] and [RZ18, Lemma 4.1].

We denote by $E \subset \mathcal{S}^{L,L'}$ be the exceptional divisor of $\mathcal{S}^{L,L'} \longrightarrow \mathcal{Q}^L$, which is a double section of $\mathcal{S}/^{L,L'}\mathbb{P}^1$ with no rational points.

Lemma V.2.2 ([SZ21, §5])**.** *Let Y be one of $\mathcal{S}^{L,L'}$ or $\mathbb{F}_n$, $n \geqslant 0$. Let $X/\mathbb{P}^1$ be a conic fibration an $\pi\colon X \longrightarrow Y$ be a birational morphism over $\mathbb{P}^1$ such that $-K_X$ is relatively ample over $\mathbb{P}^1$. Suppose that $\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\mathbb{P}^1)$ contains an element exchanging the components of at least one singular geometric fibre of X. Let $H_{\overline{\mathbf{k}}} \subseteq \mathrm{Aut}_{\overline{\mathbf{k}}}(X/\mathbb{P}^1)$ be the subgroup of elements acting trivially on $\mathrm{NS}(X_{\overline{\mathbf{k}}})$.*

1. *If $H_{\overline{\mathbf{k}}}$ is trivial, then $\mathrm{Aut}_{\overline{\mathbf{k}}}(X/\mathbb{P}^1) \simeq (\mathbb{Z}/2)^r$ for $r \in \{0,1,2\}$.*
2. *Suppose that $H_{\overline{\mathbf{k}}}$ is non-trivial.*
   - *(a) If $Y = \mathbb{F}_n$, $n \geqslant 0$, there exists $N \geqslant 1$ and a birational morphism $X \to \mathbb{F}_N$ above $\mathbb{P}^1$ blowing up $r \geqslant 1$ points $p_1,\dots,p_r$ contained in a section S_N of $\mathbb{F}_N$ with $S_N^2 = N$, contained in pairwise distinct fibres, and such that $\sum_{i=1}^r \deg(p_i) = 2N$.*
   - *(b) If $Y = \mathcal{S}^{L,L'}$, then η is the blow-up of $r \geqslant 1$ points contained in the special double section $E \subset \mathcal{S}^{L,L'}$, such that the geometric components of $p_1,\dots,p_r$ are on pairwise distinct smooth geometric fibres, and each geometric component of E contains half of the geometric components of each point.*

In (2a) the strict transforms of S_N and the exceptional section S_{-N} have both self-intersection $-N$, and they are the only sections with this property. In particular, the action of $\mathrm{Aut}(X/\mathbb{P}^1)$ and $\mathrm{Aut}_{\mathbf{k}}(X/\mathbb{P}^1)$ on these curves the induce homomorphisms $\mathrm{Aut}(X/\mathbb{P}^1) \longrightarrow \mathbb{Z}/2$ and $\mathrm{Aut}_{\mathbf{k}}(X/\mathbb{P}^1) \longrightarrow \mathbb{Z}/2$.

In (2b) the strict transform $\tilde{E}$ of E has self-intersection $\tilde{E}^2 = -2 - \sum \deg(p_i)$. Its geometric components are the only geometric sections of self-intersection$\leqslant -2$. In particular, the action of $\mathrm{Aut}(X/\mathbb{P}^1)$ and $\mathrm{Aut}_{\mathbf{k}}(X/\mathbb{P}^1)$ on the geometric components of $\tilde{E}$ induce homomorphisms $\mathrm{Aut}(X/\mathbb{P}^1) \longrightarrow \mathbb{Z}/2$ and $\mathrm{Aut}_{\mathbf{k}}(X/\mathbb{P}^1) \longrightarrow \mathbb{Z}/2$, whose kernel we denote by $\mathrm{SO}^{L,L'}$ and $\mathrm{SO}^{L,L'}(\mathbf{k})$.

Proposition V.F. *Let $X/\mathbb{P}^1$ be a conic fibration as in Lemma V.2.2(2) and $\Delta \subset \mathbb{P}^1$ the image of the set $\{p_1,\dots,p_r\}$. In each case of Proposition V.2.2(2a),(2b) there are split exact sequences:*
In (2a):

$$1 \longrightarrow T_1/\mu_N \longrightarrow \mathrm{Aut}(X/\mathbb{P}^1) \longrightarrow \mathbb{Z}/2 \to 1$$

$$1 \to \mathbf{k}^*/\mu_N(\mathbf{k}) \to \mathrm{Aut}_{\mathbf{k}}(X/\mathbb{P}^1) \to \mathbb{Z}/2 \to 1$$

where T_1 is the split one-dimensional torus and μ_N its subgroup of N-th roots of unity.
In (2b):

$$1 \longrightarrow \mathrm{SO}^{L,L'} \longrightarrow \mathrm{Aut}(X/\mathbb{P}^1) \longrightarrow \mathbb{Z}/2 \to 1$$

$$1 \to \mathrm{SO}^{L,L'}(\mathbf{k}) \to \mathrm{Aut}_{\mathbf{k}}(X/\mathbb{P}^1) \to \mathbb{Z}/2 \to 1$$

where $\mathrm{SO}^{L,L'} = \{(A,B) \in T^{L,L'} \mid AB = 1\}$, *see Proposition V.E about the torus* $T^{L,L'}$, *and* $\mathrm{SO}^{L,L'}(\mathbf{k}) \simeq \{a \in L^* \mid aa^g = 1\}$ *if* L, L' *are* $\mathbf{k}$*-isomorphicm, where* g *is the generator of* $\mathrm{Gal}(L/\mathbf{k}) \simeq \mathbb{Z}/2$*, and* $\mathrm{SO}^{L,L'}(\mathbf{k}) \simeq \mathbf{k}^*$ *if* L, L' *are not* $\mathbf{k}$*-isomorphic.*

We denote by $\Delta \subset \mathbb{P}^1$ the image of the singular fibres of $X/\mathbb{P}^1$, which is preserved by the action of $\mathrm{Aut}(X, \mathbb{P}^1)$ and of $\mathrm{Aut}_{\mathbf{k}}(X, \mathbb{P}^1)$ on $\mathbb{P}^1$.

Proposition V.G ([SZ21, §5])**.** *Let* $X/\mathbb{P}^1$ *be a conic fibration as in Lemma V.2.2(2) and* $\Delta \subset \mathbb{P}^1$ *the image of the set* $\{p_1, \dots, p_r\}$*. Then* X *is an* $\mathrm{Aut}(X, \mathbb{P}^1)$*-Mori fibre space and an* $\mathrm{Aut}_{\mathbf{k}}(X, \mathbb{P}^1)$*-Mori fibre space, and the following hold.*

1. $\mathrm{Aut}(X, \mathbb{P}^1) = \mathrm{Aut}(X)$ *if* $Y = \mathcal{S}^{L,L'}$ *and if* $Y = \mathbb{F}_N$ *and* $N \geqslant 2$.
2. *In each case of Proposition V.2.2(2a),(2b) there are exact sequences:*
   In (2a), it is split:

$$1 \longrightarrow \mathrm{Aut}(X/\mathbb{P}^1) \longrightarrow \mathrm{Aut}(X, \mathbb{P}^1) \longrightarrow \mathrm{Aut}(\mathbb{P}^1, \Delta) \longrightarrow 1$$
$$1 \to \mathrm{Aut}_{\mathbf{k}}(X/\mathbb{P}^1) \to \mathrm{Aut}_{\mathbf{k}}(X, \mathbb{P}^1) \to \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, \Delta) \to 1$$

   In (2b):

$$1 \longrightarrow \mathrm{Aut}(X/\mathbb{P}^1) \longrightarrow \mathrm{Aut}(X, \mathbb{P}^1) \longrightarrow \mathrm{Aut}(\mathbb{P}^1, \Delta) \to 1$$
$$1 \to \mathrm{Aut}_{\mathbf{k}}(X/\mathbb{P}^1) \to \mathrm{Aut}_{\mathbf{k}}(X, \mathbb{P}^1) \longrightarrow A_{\mathbf{k}} \longrightarrow 1$$

   where $A_{\mathbf{k}} = (D_{\mathbf{k}}^{L,L'} \rtimes \mathbb{Z}/2) \cap \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^1, \Delta) \subset T_1(\mathbf{k}) \rtimes \mathbb{Z}/2$*, and*
   - *if* L, L' *are* $\mathbf{k}$*-isomorphic:* $D_{\mathbf{k}}^{L,L} \simeq \{\mu \in \mathbf{k}^* \mid \mu = \lambda\lambda^g, \lambda \in L^*\}$*, where* g *is the generator of* $\mathrm{Gal}(L/\mathbf{k})$*,*
   - *if* L, L' *are not* $\mathbf{k}$*-isomorphic:* $D_{\mathbf{k}}^{L,L'} = \{\lambda\lambda^{gg'} \in F \mid \lambda \in K, \lambda\lambda^g = 1\}$*, where* $\mathbf{k} \subset F \subset LL'$ *is the intermediate extension such that* $\mathrm{Gal}(F/\mathbf{k}) \simeq \langle gg' \rangle \subset \mathrm{Gal}(K/L) \times \mathrm{Gal}(L/\mathbf{k})$ *where* g, g' *are the generators of* $\mathrm{Gal}(K/L), \mathrm{Gal}(L/\mathbf{k})$*, respectively.*

For $\mathbf{k} = \mathbb{R}$ we have $D_{\mathbb{R}}^{\mathbb{C},\mathbb{C}} = \mathbb{R}_{>0}$ as in [RZ18]. In the situation of Lemma V.2.2(2a) with $N = 1$, the surface X is a del Pezzo surface of degree 6 and $\mathrm{Aut}(X, \mathbb{P}^1) \subsetneq \mathrm{Aut}(X)$ and $\mathrm{Aut}_{\mathbf{k}}(X, \mathbb{P}^1) \subseteq \mathrm{Aut}_{\mathbf{k}}(X)$ [SZ21, §5]. The exact sequence in (2b) is split if any element in F is a square. We do not know whether it is split in general.

V.3 The classification

We can now describe the classification of the infinite algebraic groups acting birationally on $\mathbb{P}^2$ up to inclusion and conjugation. It generalises the classification of infinite algebraic groups acting on $\mathbb{P}^2$ over $\mathbb{C}$ from [Bla09b] and over $\mathbb{R}$ from [RZ18] and holds no surprises, except perhaps over the field with two elements. From Proposition V.A and the propositions in §V.2, we obtain the following statement.

Theorem V.H ([SZ21, Theorem 1.1])**.** *Let* G *an infinite algebraic group acting birationally on* $\mathbb{P}^2$*. Then there is a birational map* $\mathbb{P}^2 \dashrightarrow X$ *that conjugates* G *to a subgroup of* $\mathrm{Aut}(X)$ *or* $\mathrm{Aut}(X, \mathbb{P}^1)$*, with* X *one of the following surfaces:*

1. *$X = \mathbb{P}^2$, $X = \mathcal{Q}^L$, $X = \mathbb{F}_n$, $n \neq 1$;*
2. *X is the blow-up of $\mathbb{P}^2$ in three rational non-collinear points;*
3. *X is a del Pezzo surface of degree 6 with $\mathrm{rkNS}(X) = 1$ as in Proposition V.C(1).*
4. *X is a del Pezzo surface of degree 6 with $\mathrm{rkNS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 1$ not as in (2) or (3);*
5. *X is a del Pezzo surface of degree 6 with $\mathrm{rkPic}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 2$;*
6. *there is a conic fibration $\pi\colon X \longrightarrow \mathbb{P}^1$ as in Lemma V.2.2(2a) with $N \geqslant 2$ and $\mathrm{rkNS}(X_{\overline{\mathbf{k}}})^{\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\mathbb{P}^1)} = \mathrm{rkNS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X,\mathbb{P}^1)} = 2$.*
7. *there is a conic fibration $\pi\colon X \longrightarrow \mathbb{P}^1$ as in Lemma V.2.2(2b) and $\mathrm{rkNS}(X_{\overline{\mathbf{k}}})^{\mathrm{Aut}_{\overline{\mathbf{k}}}(X,\mathbb{P}^1)} = \mathrm{rkNS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X,\mathbb{P}^1)} = 2$.*

Next we classify the groups in Theorem V.H up to conjugation by birational maps. Let G be an affine group and X/B a G-Mori fibre space. We call it *G-birationally rigid* if for any G-equivariant birational map $\varphi\colon X \dashrightarrow X'$ to another G-Mori fibre space X'/B' we have $X' \simeq X$. In particular we have $\varphi\,\mathrm{Aut}(X)\varphi^{-1} = \mathrm{Aut}(X')$. Note that we do not ask φ to be an isomorphism. We call it *G-birationally superrigid* if any G-equivariant birational map $X \dashrightarrow X'$ to another G-Mori fibre space X'/B' is an isomorphism. If we replace G by $G(\mathbf{k})$ everywhere, we get the notion of $G(\mathbf{k})$-birationally rigid and $G(\mathbf{k})$-birationally superrigid.

Note that G-birationally (super)rigid does not imply $G(\mathbf{k})$-birationally (super)rigid. Indeed, a rational del Pezzo surface X of degree 6 with $\mathrm{rkNS}(X)^{\mathrm{Aut}_{\mathbf{k}}(X)} = 2$ is never $\mathrm{Aut}_{\mathbf{k}}(X)$-birationally rigid because there is an $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant birational contraction to some $\mathcal{Q}^L$ by Proposition V.E. However, it is $\mathrm{Aut}(X)$-birationally superrigid by the next theorem.

Theorem V.I ([SZ21, Theorem 1.2])**.** *Any surface X from the list in Theorem V.H is $\mathrm{Aut}(X)$-birationally superrigid. Moreover, the following hold:*

1. *If X is as in (1),(4) or (7), it is $\mathrm{Aut}_{\mathbf{k}}(X)$-birationally superrigid.*
2. *Suppose that X is as in (2) or (3).*
   If $|\mathbf{k}| \geqslant 3$, then X is $\mathrm{Aut}_{\mathbf{k}}(X)$-birationally superrigid.
   If $|\mathbf{k}| = 2$, there are $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant birational maps $X \dashrightarrow X'$, where X' is a del Pezzo surface of degree 6 as in Proposition V.D(2).
3. *Any conic fibration $X/\mathbb{P}^1$ from (6) is $\mathrm{Aut}_{\mathbf{k}}(X)$-birationally superrigid if $\mathbf{k}^*/\mu_n(\mathbf{k})$ is non-trivial. If $\mathbf{k}^*/\mu_n(\mathbf{k})$ is trivial, and $X \dashrightarrow Y$ a $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant birational map to a surface Y from Theorem V.H, then $X \simeq Y$.*

Theorem V.I is proven by studying all possible options of $\mathrm{Aut}_{\mathbf{k}}(X)$-equivariant Sarkisov links starting from an $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre space X in Theorem V.H.

We say that an algebraic group G acting birationaly on $\mathbb{P}^2$ is *maximal* if it is maximal with respect to inclusion. We say that $G(\mathbf{k})$ is *maximal* if for any algebraic subgroup G' of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ containing $G(\mathbf{k})$, we have $G(\mathbf{k}) = G'(\mathbf{k})$. For instance, if $|\mathbf{k}| = 2$ and X is the blow-up of $\mathbb{P}^2$ in three non-collinear rational points, then $\mathrm{Aut}_{\mathbf{k}}(X)$ is not maximal by Theorem V.I.

Corollary V.J ([SZ21, Corollary 1.3])**.** *Let H an infinite algebraic subgroup of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$.*

1. *Then H is contained in a maximal algebraic subgroup G of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$.*

2. *Up to conjugation by a birational map, the maximal infinite algebraic subgroups of* $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ *are precisely the groups* $\mathrm{Aut}(X)$ *in Theorem V.H. Two maximal infinite subgroups* $\mathrm{Aut}(X)$ *and* $\mathrm{Aut}(X')$ *are conjugate by a biratonal map if and only if* $X \simeq X'$.
3. $H(\mathbf{k})$ *is maximal if and only if it is conjugate to one of the* $\mathrm{Aut}_{\mathbf{k}}(X)$ *from (1), (4), (6), (7), and from (2), (3) if* $|\mathbf{k}| \geqslant 3$. *Two such groups* $\mathrm{Aut}_{\mathbf{k}}(X)$ *and* $\mathrm{Aut}_{\mathbf{k}}(X')$ *are conjugate by a birational map if and only if* $X \simeq X'$.

We can also describe the parameter space of each family of conjugacy classes of the $\mathrm{Aut}_{\mathbf{k}}(X)$-Mori fibre spaces in Theorem V.H, see [SZ21, Theorem 1.4]. For instance, for the del Pezzo surfaces of degree 6, they correspond to the $\mathbf{k}$-isomorphism classes of the associated finite field extensions of $\mathbf{k}$.

This concludes the classification of infinite algebraic groups acting birationally on $\mathbb{P}^2_{\mathbf{k}}$ over a perfect field $\mathbf{k}$ up to conjugation and inclusion.

A Appendix: the classification of elementary relations in dimension 2

Throughout this section, $\mathbf{k}$ is a perfect field. In order to give the complete list of elementary relations in $\mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}})$, one needs to classify the rational rank 3 fibrations in dimension 2 and all possible contractions from each of them, see Example II.1.9. Accordingly, an elementary relation will be presented in the form of a commutative blow-up diagram with the dominating rank 3 fibration in the center. A rank 1 fibration will be denoted by X_i/B_i, a rank 2 fibration by Y_i/B_i and the dominating rank 3 fibration by T/B. The arrows $\longrightarrow$ in the diagram are birational contractions and the dashed arrows $\dashrightarrow$ (with or without head) are the Sarkisov links appearing in the elementary relation.

We will first look at the case $T/\mathbb{P}^1$ and then at T/pt, where we will order the cases according to the degree K_T^2.

A.1 Elementary relations above a curve

We classify the elementary relations dominated by a rational rank 3 fibration $T/\mathbb{P}^1$.

Remark A.1.1. Theorem II.3.4 (or Remark II.3.5) imply that the elementary relations dominated by a rank 3 fibration $T/\mathbb{P}^1$ involves only Sarkisov links between Mori conic bundles. Any birational morphism $T \longrightarrow X \longrightarrow \mathbb{P}^1$, where $X/\mathbb{P}^1$ is a Mori conic bundle and $T/\mathbb{P}^1$ is a rank 3 fibration, is the blow up of two closed points p, q and the geometric components of p and q contained in pairwise distinct smooth geometric fibres. The only contractions from T over $\mathbb{P}^1$ are the contractions of the exceptional divisors E_p and E_q of p and q, respectively, and of the strict transform f_p and f_q of the fibre through p and q, respectively.

The classification of rational Mori conic bundles in Lemma III.1.6 implies that any elementary relation in $\mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}})$ dominated by a rank 3 fibration $T/\mathbb{P}^1$ is one of the following three (see Definition III.1.5 for $\mathcal{X}/\mathbb{P}^1$ and $\mathcal{S}/\mathbb{P}^1$).

$$
\begin{tikzcd}
& & \mathbb{F}_l/\mathbb{P}^1 \arrow[ddll, dashed, no head] \arrow[ddrr, dashed, no head] & & \\
& Y_2/\mathbb{P}^1 \arrow[ur, "f_q"] \arrow[dl, "E_q"'] & & Y_2'/\mathbb{P}^1 \arrow[ul, "f_p"'] \arrow[dr, "E_p"] & \\
\mathbb{F}_m/\mathbb{P}^1 \arrow[ddrr, dashed, no head] & & T/\mathbb{P}^1 \arrow[ul, "f_p"] \arrow[ur, "f_q"'] \arrow[dl, "E_q"] \arrow[dr, "E_p"'] & & \mathbb{F}_k/\mathbb{P}^1 \arrow[ddll, dashed, no head] \\
& Y_1/\mathbb{P}^1 \arrow[ul, "f_p"] \arrow[dr, "E_p"'] & & Y_1'/\mathbb{P}^1 \arrow[ur, "f_q"'] \arrow[dl, "E_q"] & \\
& & \mathbb{F}_n/\mathbb{P}^1 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \mathcal{S}/\mathbb{P}^1 \arrow[ddll, dashed, no head] \arrow[ddrr, dashed, no head] & & \\
& Y_2/\mathbb{P}^1 \arrow[ur, "f_q"] \arrow[dl, "E_q"'] & & Y_2'/\mathbb{P}^1 \arrow[ul, "f_p"'] \arrow[dr, "E_p"] & \\
\mathcal{S}/\mathbb{P}^1 \arrow[ddrr, dashed, no head] & & T/\mathbb{P}^1 \arrow[ul, "f_p"] \arrow[ur, "f_q"'] \arrow[dl, "E_q"] \arrow[dr, "E_p"'] & & \mathcal{S}/\mathbb{P}^1 \arrow[ddll, dashed, no head] \\
& Y_1/\mathbb{P}^1 \arrow[ul, "f_p"] \arrow[dr, "E_p"'] & & Y_1'/\mathbb{P}^1 \arrow[ur, "f_q"'] \arrow[dl, "E_q"] & \\
& & \mathcal{S}/\mathbb{P}^1 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & \mathcal{X}/\mathbb{P}^1 \arrow[ddll, dashed, no head] \arrow[ddrr, dashed, no head] & & \\
& Y_2/\mathbb{P}^1 \arrow[ur, "f_q"] \arrow[dl, "E_q"'] & & Y_2'/\mathbb{P}^1 \arrow[ul, "f_p"'] \arrow[dr, "E_p"] & \\
\mathcal{X}/\mathbb{P}^1 \arrow[ddrr, dashed, no head] & & T/\mathbb{P}^1 \arrow[ul, "f_p"] \arrow[ur, "f_q"'] \arrow[dl, "E_q"] \arrow[dr, "E_p"'] & & \mathcal{X}/\mathbb{P}^1 \arrow[ddll, dashed, no head] \\
& Y_1/\mathbb{P}^1 \arrow[ul, "f_p"] \arrow[dr, "E_p"'] & & Y_1'/\mathbb{P}^1 \arrow[ur, "f_q"'] \arrow[dl, "E_q"] & \\
& & \mathcal{X}/\mathbb{P}^1 & &
\end{tikzcd}
$$

A.2 Elementary relations dominated by a del Pezzo surface

We now list the elementary relations dominated by a rank 3 fibration T/pt, where T is rational. The list is ordered according to the degree K_T^2 of T, starting with $K_T^2 = 1$ and working our way up to $K_T^2 = 7$. According to the argument in Remark II.3.5, T/pt factors through some Mori fibre space X_1/pt, that is a del Pezzo surface X_1 with $\rho(X_1) = 1$, or it dominates a link of type IV.

In the first case, T is the blow of X_1 in two points p and p' of degree $\deg(p) = d$ and $\deg(p') = d'$ with $d + d' = K_T^2 - K_{X_1}^2$. We list for each T the cases (X_1, d, d'), and each time we start with $X_1 = \mathbb{P}^2$, then $X_1 = \mathcal{Q}$ and then, if not covered yet, X_1 a del Pezzo surface of degree 6 with $\rho(X_1) = 1$. The case where X_1 is a del Pezzo surface of degree 5 with $\rho(X_1) = 1$ will always turn out to be covered by the other three.

In the second case, T is the blow-up in one point p of degree d of a conic bundle $X_1/\mathbb{P}^1$ with $\rho(X_1/\mathbb{P}^1) = 1$ that is also a del Pezzo surface. It follows from [Sch19, Remark 6.1, Lemma 6.13] that X_1 is isomorphic to $\mathbb{F}_0$ or $\mathbb{F}_1$, or to a del Pezzo surface of degree 6 or of degree 5. By assumption, there is a Sarkisov link of type IV starting from X_1, so $X_1 = \mathbb{F}_0$ according to [Isk96, Theorem 2.6]. If the point p is of odd degree d, then the diagram will show up as diagram of case $(\mathbb{P}^2, 1, d)$ from above. Indeed, the blow-up of $F_0/\mathbb{P}^1$ of p induces a Sarkisov link $\mathbb{F}_0 \dashrightarrow \mathbb{F}_1$ of type II. The birational contraction $\mathbb{F}_1 \longrightarrow \mathbb{P}^2$ will be included in the diagram, so $T \longrightarrow \mathbb{F}_1 \longrightarrow \mathbb{P}^2$ is included in the diagram. In conclusion, we only have to list the cases $(\mathbb{F}_0, d)$ with d even.

Before we start, let us introduce *Geiser links*: Let X_1/pt be a rational Mori fibre space and suppose that it contains a point p of degree $K_{X_1}^2 - 2$ such that its blowup $T' \longrightarrow X_1$ yields a del Pezzo surface T', which has degree $K_{T'}^2 = 2$. Then T'/pt is a rank 2 fibration and dominates a Sarkisov $\chi\colon X_1 \dashrightarrow X_1$ called *Geiser link*. Geometrically, it is defined as follows: $|-K_{T'}|$ induces a double cover $T' \longrightarrow \mathbb{P}^2$ ramified over a smooth plane quartic curve. The Galois involution of the double cover induces a birational involution $\gamma\colon X_1 \dashrightarrow X_1$, called *Geiser involution*, whose base-locus is p. There exists $\alpha \in \mathrm{Aut}(X_1)$ such that $\chi \circ \alpha = \gamma$.

A.2.1 Elementary relations dominated by a del Pezzo surface of degree 1

Suppose that T/pt is a rank 3 fibration with $K_T^2 = 1$. There is a birational morphism $T_{\overline{\mathbf{k}}} \longrightarrow \mathbb{P}^2_{\overline{\mathbf{k}}}$ blowing up eight points $p_1, \ldots, p_8$ and $T_{\overline{\mathbf{k}}}$ contains precisely 240 (-1)-curves:

- the eight exceptional divisors,
- the 28 strict transform of the lines through two of the p_i,
- the 56 conics passing through five of $p_1, \ldots, p_8$,
- the 56 cubics passing through seven of the p_i and singular at one of these seven,
- the 56 quartics passing through $p_1, \ldots, p_8$, three of the p_i its double points
- the 28 quintics passing through $p_1, \ldots, p_8$, six of the p_i its double points,
- the eight sextics passing through $p_1, \ldots, p_8$, one of the p_i a triple point and double points at the remaining seven.

The case $(\mathbb{P}^2, 1, 7)$:

Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a rational point and a point of degree 7, and denote respectively by $E \subset T$ and $E_1', \ldots, E_7' \subset T$ the geometric components of their exceptional divisors. Let $L \subset T$ be the pullback of a general line in $\mathbb{P}^2$. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves on $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 8$ are the following:

$$
\begin{aligned}
&E, \quad E_1', \ldots, E_7', \qquad \ell_i := L - E - E_i' \\
&D_{i_7} := 3L - 2E - E_{i_1}' - \cdots - E_{i_6}' \\
&D_{i_1}' := 3L - 2E_{i_1}' - E_{i_2}' - \cdots - E_{i_7}' \\
&Q_{i_1} := 5L - E - E_{i_1}' - 2E_{i_2}' - \cdots - 2E_{i_7}' \\
&S := 6L - 3E - 2E_1' - \cdots - 2E_7', \\
&S_{i_1}' := 6L - 3E_{i_1}' - 2E_{i_2}' - \cdots - 2E_{i_7}' - 2E.
\end{aligned}
$$

$$
\begin{tikzcd}
 & \mathbb{F}_1/\mathbb{P}^1 \arrow[r, dashed, no head] \arrow[dl, dashed, no head, bend right] & \mathbb{P}^2/\mathrm{pt} \arrow[rr, dashed, no head] & & \mathbb{P}^2/\mathrm{pt} \arrow[r, dashed, no head] & \mathbb{F}_1/\mathbb{P}^1 \arrow[dr, dashed, no head, bend left] & \\
\mathbb{F}_0/\mathbb{P}^1 \arrow[dd, no head, "\simeq"] & T/\mathbb{P}^1 \arrow[l, "\ell"'] \arrow[u, "D"'] & \mathbb{F}_1/\mathrm{pt} \arrow[ul] \arrow[u, "S"] & Y_2'/\mathrm{pt} \arrow[ul, "D"'] \arrow[ur, "S'"] & \mathbb{F}_1/\mathrm{pt} \arrow[u, "S"] \arrow[ur] & T/\mathbb{P}^1 \arrow[r, "Q"] \arrow[u, "S'"] & \mathbb{F}_0/\mathbb{P}^1 \arrow[dd, no head, "\simeq"'] \\
 & \mathbb{F}_0/\mathrm{pt} \arrow[ul] \arrow[dl] & & T/\mathrm{pt} \arrow[ll, "\ell"] \arrow[rr, "Q"] \arrow[ull] \arrow[ul, "D"'] \arrow[u, "S"] \arrow[ur, "S'"] \arrow[urr] \arrow[dll] \arrow[dl, "E'"'] \arrow[d, "E"] \arrow[dr, "D'"] \arrow[drr] & & \mathbb{F}_0/\mathrm{pt} \arrow[ur] \arrow[dr] & \\
\mathbb{F}_0/\mathbb{P}^1 \arrow[dr, dashed, no head, bend right] & T/\mathbb{P}^1 \arrow[l, "\ell"] \arrow[d, "E'"] & \mathbb{F}_1/\mathrm{pt} \arrow[dl] \arrow[d, "E"] & Y_2/\mathrm{pt} \arrow[dl, "E'"'] \arrow[dr, "D'"] & \mathbb{F}_1/\mathrm{pt} \arrow[d, "E"] \arrow[dr] & T/\mathbb{P}^1 \arrow[r, "Q"'] \arrow[d, "D'"] & \mathbb{F}_0/\mathbb{P}^1 \arrow[dl, dashed, no head, bend left] \\
 & \mathbb{F}_1/\mathbb{P}^1 \arrow[r, dashed, no head] & \mathbb{P}^2/\mathrm{pt} \arrow[rr, dashed, no head, "\gamma"] & & \mathbb{P}^2/\mathrm{pt} \arrow[r, dashed, no head] & \mathbb{F}_1/\mathbb{P}^1 &
\end{tikzcd}
$$

We have

$$E \cdot E_i' = E \cdot D_i' = D_i \cdot S = S \cdot S_i' = 0, \quad E_i' \cdot \ell_j = \ell_i \cdot D_i = D_i' \cdot Q_j = Q_i \cdot S_j' = \delta_{ij}, \quad E_i' \cdot D_j = 1 - \delta_{ij}$$

for all i, j, where δ_{ij} is the Kronecker delta. All other pairs of orbits have no trivial intersections. Completing the contraction diagram, we obtain the commutative diagram above, where by E' we mean $E_1' + \cdots + E_7'$ and so forth.

The case $(X_1, d_1, d_2) = (\mathbb{P}^2, 2, 6)$:

Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a point of degree 2 and a point of degree 6, and denote respectively by E_1, E_2 and $E'_1, \ldots, E'_6$ the geometric components of their exceptional divisors. Let L be the pullback of a general line in T. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 8$ with pairwise disjoint members are the following:

$$\begin{aligned}
&E_1, E_2, \quad E'_1, \ldots, E'_6, \quad \ell := L - E_1 - E_2\\
&C_{i_6} = 2L - E'_{i_1} - \cdots E'_{i_5}\\
&F_{i_1} := 4L - 2E_1 - 2E_2 - 2E'_{i_1} - E'_{i_2} - \cdots - E'_{i_6}\\
&Q := 5L - E_1 - E_2 - 2E'_1 - \cdots - 2E'_6\\
&S_i := 6L - 3E_i - 2E_{3-i} - 2E'_1 - \cdots - 2E'_6\\
&S'_{i_1} := 6L - 3E'_{i_1} - 2E_1 - 2E_2 - 2E'_{i_2} - \cdots - 2E'_{i_6}.
\end{aligned}$$

$$\begin{tikzcd}
 & \mathbb{P}^2/\mathrm{pt} \arrow[rr, dash, dashed] \arrow[dl, dash, dashed] & & \mathbb{P}^2/\mathrm{pt} \arrow[dr, dash, dashed] & \\
\mathcal{Q}/\mathrm{pt} \arrow[dd, dash, dashed] & Y_3/\mathrm{pt} \arrow[u, "S"] \arrow[l, "Q"'] & Y'_3/\mathrm{pt} \arrow[ul, "S'"] \arrow[ur, "F"] & Y''_3/\mathrm{pt} \arrow[u, "S"] \arrow[r, "\ell"] & \mathcal{Q}/\mathrm{pt} \arrow[dd, dash, dashed] \\
 & Y_2/\mathrm{pt} \arrow[ul, "S'"] \arrow[dl, "C"] & T/\mathrm{pt} \arrow[ul, "S'"] \arrow[u, "S"] \arrow[ur, "F"] \arrow[l, "Q"'] \arrow[r, "\ell"] \arrow[dl, "C"] \arrow[d, "E"] \arrow[dr, "E'"] & Y'_2/\mathrm{pt} \arrow[ur, "F"] \arrow[dr, "E'"] & \\
\mathcal{Q}/\mathrm{pt} \arrow[dr, dash, dashed] & Y_1/\mathrm{pt} \arrow[l, "Q"'] \arrow[d, "E"] & Y'_1/\mathrm{pt} \arrow[dl, "C"] \arrow[dr, "E'"] & Y''_1/\mathrm{pt} \arrow[r, "\ell"] \arrow[d, "E"] & \mathcal{Q}/\mathrm{pt} \arrow[dl, dash, dashed] \\
 & \mathbb{P}^2 \arrow[rr, dash, dashed] & & \mathbb{P}^2 &
\end{tikzcd}$$

We have

$$E_i{\cdot}E'_j = E_i{\cdot}C_j = E'_i{\cdot}\ell = \ell{\cdot}F_i = C_i{\cdot}Q = F_i{\cdot}S_j = Q{\cdot}S'_i = S_i{\cdot}S'_j = 0, \quad E'_i{\cdot}C_j = \delta_j, \quad F_i{\cdot}S'_j = 1-\delta_{ij}$$

for all i, j. All other pair of orbits have no trivial intersections. This yields again all possible contractions from T to a rank 2 fibration and which ones commute. We the commutative diagram above, where all quadrics $\mathcal{Q}$ are isomorphic according to Lemma V.B.

The case $(\mathbb{P}^2, 3, 5)$:
Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a point of degree 3 and a point of degree 5, and denote respectively by E_1, E_2, E_3 and $E'_1, \ldots, E'_5$ the geometric components of their exceptional divisors. Let $L \subset T$ be the pullback of a general line in $\mathbb{P}^2$. Among the 240 (-1)-curves on $T_{\overline{\mathbf{k}}}$, the only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of cardinality $\leqslant 8$ are the following:

$$\begin{aligned}
&E_1, E_2, E_3, \quad E'_1, \ldots, E'_5,\\
&\ell_i := L - E_1 - E_2 - E_3 - E'_i\\
&C := 2L - E'_1 - \cdots - E'_5\\
&D_{i_1} := 3L - 2E_{i_1} - E_{i_2} - E'_1 - \cdots - E'_5\\
&F := 4L - 2E_1 - 2E_2 - 2E_3 - E'_1 - \cdots - E'_5\\
&Q_{i_3} := 5L - E_{i_1} - E_{i_2} - 2E_{i_3} - 2E'_1 - \cdots - 2E'_5\\
&S_{i_1} := 6L - 3E_{i_1} - 2E_{i_2} - 2E_{i_3} - 2E'_1 - \cdots - 2E'_5\\
&S'_{i_1} := 6L - 3E'_{i_1} - 2E'_{i_2} - \cdots - 2E'_{i_5} - 2E_1 - 2E_2 - 2E_3
\end{aligned}$$

$$\begin{tikzcd}
 & \mathbb{P}^2/\mathrm{pt} \arrow[rr, dash, dashed] \arrow[dl, dash, dashed] & & \mathbb{P}^2/\mathrm{pt} \arrow[dr, dash, dashed] & \\
X_5/\mathrm{pt} \arrow[dd, dash, dashed] & Y_3/\mathrm{pt} \arrow[u, "S'"] \arrow[l, "F"'] & Y'_3/\mathrm{pt} \arrow[ul, "S"] \arrow[ur, "Q"] & Y''_3/\mathrm{pt} \arrow[u, "S'"] \arrow[r, "C"] & X_5/\mathrm{pt} \arrow[dd, dash, dashed] \\
 & Y_2/\mathrm{pt} \arrow[ul, "S"] \arrow[dl, "\ell"] & T/\mathrm{pt} \arrow[ul, "S"] \arrow[u, "S'"] \arrow[ur, "Q"] \arrow[l, "F"'] \arrow[r, "C"] \arrow[dl, "\ell"] \arrow[d, "E'"] \arrow[dr, "E"] & Y'_2/\mathrm{pt} \arrow[ur, "Q"] \arrow[dr, "E"] & \\
X_5/\mathrm{pt} \arrow[dr, dash, dashed] & Y_1/\mathrm{pt} \arrow[l, "F"'] \arrow[d, "E'"] & Y'_1/\mathrm{pt} \arrow[dl, "\ell"] \arrow[dr, "E"] & Y''_1/\mathrm{pt} \arrow[r, "C"] \arrow[d, "E'"] & X_5/\mathrm{pt} \arrow[dl, dash, dashed] \\
 & \mathbb{P}^2/\mathrm{pt} \arrow[rr, dash, dashed] & & \mathbb{P}^2/\mathrm{pt} &
\end{tikzcd}$$

The orbit $D_1, \cdots, D_6$ has intersecting members, so it cannot be contracted from T. We have

$$E_i \cdot E'_j = E_i \cdot C = E'_j \cdot \ell_i = C \cdot Q_i = F \cdot S_i = F \cdot \ell = S_i \cdot S'_j = Q \cdot S'_j = 0$$

for any $i = 1, 2, 3$, $j = 1, \ldots, 5$. This yields the commutative diagram above, where X_5 is a del Pezzo surface of degree 5. Two del Pezzo surfaces of degree 5 joined by a link are joined by a Geiser link, hence the two surfaces are isomorphic. Any two points of degree 3 not contained in a line can be sent onto one-another by an automorphism of

$\mathbb{P}^2$ [Sch19, Lemma 6.10]. So, the two links $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ in the diagram are involutions up to pre-composing with an automorphism of $\mathbb{P}^2$. It follows that all del Pezzo surfaces of degree 5 in the diagram are isomorphic.

<u>The case $(\mathbb{P}^2, 4, 4)$:</u>
Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of two points of degree 4 and denote respectively by $E_1, \ldots, E_4$ and $E_1', \ldots, E_4'$ the geometric components of their exceptional divisors. Let $L \subset \mathbb{P}^2$ be a the pullback of a general line in $\mathbb{P}^2$. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ orbits of (-1)-curves on $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 8$ with pairwise disjoint members are the following:

$$\begin{aligned}
&E_1, \ldots, E_4, \quad E_1', \ldots, E_4' \\
&C_i := 2L - E_1 - \cdots - E_4 - E_i', \quad C_i' := 2L - E_1' - \cdots - E_4' - E_i \\
&F_{i_4} := 4L - 2E_{i_1} - 2E_{i_2} - 2E_{i_3} - E_{i_4} - E_1' - \cdots - E_4', \\
&F_{i_4}' := 4L - 2E_{i_1}' - 2E_{i_2}' - 2E_{i_3}' - E_{i_4} - E_1 - \cdots - E_4, \\
&S_i := 6L - 3E_{i_1} - 2E_{i_2} - 2E_{i_3} - 2E_{i_4} - 2E_1' - \cdots - 2E_4', \\
&S_i' := 6L - 3E_{i_1}' - 2E_{i_2}' - 2E_{i_3}' - 2E_{i_4}' - 2E_1 - \cdots - 2E_4.
\end{aligned}$$

We have

$$E_i \cdot E_j' = C_i \cdot F_j' = C_i' \cdot F_j = S_i \cdot S_j' = 0, \quad F_i \cdot S_j = F_i' \cdot F_j = \delta_{ij}$$

for all i, j. No other pairs have trivial intersections. Completing the blow-up diagram, we obtain the commutative diagram in Figure A.1.

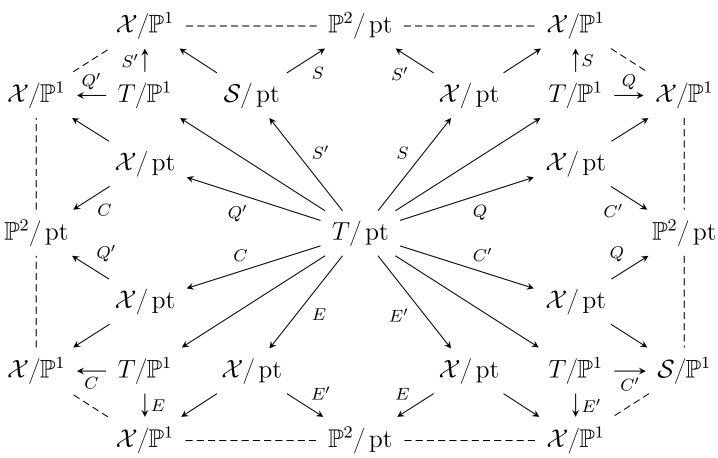


Figure A.1: the case $(\mathbb{P}^2, 4, 4)$.

Remark A.2.1. We want to express the (-1)-curves on $T_{\overline{\mathbf{k}}}$ in terms of curves and exceptional divisors with respect to a birational morphism $T_{\overline{\mathbf{k}}} \longrightarrow \mathcal{Q}_{\overline{\mathbf{k}}}$ blowing up seven points. For this, consider the blow-up $T_{\overline{\mathbf{k}}} \longrightarrow \mathbb{P}^2_{\overline{\mathbf{k}}}$ of points $p_1, p_2, q_2, \ldots, q_7$. Let $\mathbb{P}^2_{\overline{\mathbf{k}}} \dashrightarrow \mathcal{Q}_{\overline{\mathbf{k}}}$ be the composition of the blow-up of p_1, p_2 and the contraction of the line ℓ through p_1, p_2 onto a point $q_1 \in \mathcal{Q}_{\overline{\mathbf{k}}}$. It induces a birational morphism $T_{\overline{\mathbf{k}}} \longrightarrow \mathcal{Q}_{\overline{\mathbf{k}}}$ blowing up $q_1, q_2, \ldots, q_7$. Looking from $\mathcal{Q}_{\overline{\mathbf{k}}}$, the (-1)-curves in $T_{\overline{\mathbf{k}}}$ are the strict transform of the following curves:

- seven exceptional divisors,
- for each fibration the seven fibres through one q_i,
- the 35 of bidegree $(1,1)$ through three of the q_i,
- the 42 of bidegree $(1,2)$ and $(2,1)$ through five of the q_i,
- the 42 of bidegree $(2,2)$ through six of the q_i having a double point at one of them,
- the 2 of bidegree $(1,3)$ and $(3,1)$ through $q_1,\dots,q_7$,
- the 42 of bidegree $(2,3)$ and $(3,2)$ through $q_1,\dots,q_7$ with a double point at two of the q_i,
- the 35 of bidegree $(3,3)$ through $q_1,\dots,q_7$ with a double point at four of the q_i,
- the 14 of bidegree $(3,4)$ and $(4,3)$ through $q_1,\dots,q_7$ with double points at six of the q_i,
- the seven of bidegree $(4,4)$ through $q_1,\dots,q_7$ with a triple point at of the q_i and double points at the remaining six q_i.

If F is the pullback of the curve generating $\mathrm{NS}(\mathcal{Q})$, then the above list translates as follows (keeping the order), where the number of geometric components of the curve are indicated in parenthesis.

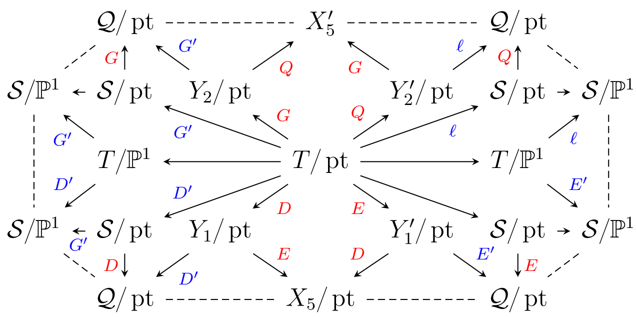
seven exceptional divisors E_i (1)	$4F-2E_1-\dots-2E_7$ (2)
$F-2E_i$ (1)	$5F-4E_{i_1}-4E_{i_2}-E_{i_3}-\dots-E_{i_7}$ (2)
$F-E_{i_1}-E_{i_2}-E_{i_3}$ (1)	$3F-2E_{i_1}-\dots-2E_{i_4}-E_{i_5}-E_{i_6}-E_{i_7}$ (1)
$3F-2E_{i_1}-\dots-2E_{i_5}$ (2)	$7F-2E_{i_1}-4E_{i_2}-\dots-4E_{i_7}$ (2)
$2F-2E_{i_1}-E_{i_2}-\dots-E_{i_6}$ (1)	$4F-3E_{i_1}-2E_{i_2}-\dots-2E_{i_7}$ (1)

Since we have already covered all birational contractions $T \longrightarrow \mathbb{P}^2$, we only need to consider contractions of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves with $\leqslant 7$ members.

The case $(\mathcal{Q},2,5)$:

Let $T \longrightarrow \mathcal{Q}$ be the blow-up of a point of degree 2 and a point of degree 5, and denote respectively by E_1, E_2 and $E'_1,\dots,E'_5$ the geometric components of their exceptional divisors. Let $F \subset T$ be the pullback of the curve generating $\mathrm{NS}(\mathcal{Q})$. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves on $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 7$ and with pairwise disjoint members are the following, where the number of geometric components is indicated in parenthesis.

E_1, E_2, $\quad E'_1,\dots,E'_5$,
$\ell_i := F-E_1-E_2-E'_i$ (1)
$D := 3F-2E'_1-\dots-2E'_2$ (2)
$D'_{i_5} := 3F-2E'_{i_1}-\dots-2E'_{i_4}-E'_{i_5}-E_1-E_1$ (1)
$G_i := 4F-3E_i-2E_{3-i}-2E'_1-\dots-2E'_5$ (1)
$G'_{i_1} := 4F-3E'_{i_1}-2E_1-2E_2-2E'_{i_2}-\dots-2E'_{i_5}$ (1)
$Q := 5F-4E_1-4E_2-2E'_1-\dots-2E'_5,$ (2)



We have

$$E_i \cdot E'_j = E_i \cdot D = D \cdot D'_i = Q \cdot \ell_i = Q \cdot G_i = G_i \cdot G'_j = 0, \quad E'_i \cdot \ell_j = D'_i \cdot G'_j = \delta_{ij}$$

for all i,j. All other pairs have no trivial intersections. Completing the commutative diagram yields the commutative diagram above, where X_5 and X'_5 are del Pezzo surfaces of degree 5.

The case $(\mathcal{Q}, 3, 4)$:

Let $T \longrightarrow \mathcal{Q}$ be the blow-up of a point of degree 3 and a point of degree 4, and denote respectively by E_1, E_2, E_3 and $E'_1, \dots, E'_4$ the geometric components of their exceptional divisors. Let $F \subset T$ be the pullback of the generator of $\mathrm{NS}(\mathcal{Q})$. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves on $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 7$ and pairwise disjoint members are the following, where the number of geometric components is indicated in parenthesis.

$E_1, E_2, E_3, \quad E'_1, \dots, E'_4,$

$\ell := F - E_1 - E_2 - E_3$ (1)

$\ell'_{i_4} := F - E'_{i_1} - \cdots - E'_{i_3}$ (1)

$D := 3F - 2E'_1 - \cdots - 2E'_4 - E_1 - E_2 - E_3$ (1)

$$\begin{tikzcd}
 & \mathcal{Q}/\mathrm{pt} \arrow[rr, dashed, no head] \arrow[dl, dashed, no head] & & \mathcal{Q}/\mathrm{pt} \arrow[dr, dashed, no head] & \\
X'_6/\mathrm{pt} \arrow[dd, dashed, no head] & Y_3/\mathrm{pt} \arrow[u, "G"] \arrow[l, "\ell"'] & Y'_3/\mathrm{pt} \arrow[ul, "D'"'] \arrow[ur, "G'"] & Y''_3/\mathrm{pt} \arrow[u, "G"'] \arrow[r, "D"] & X_6/\mathrm{pt} \arrow[dd, dashed, no head] \\
 & Y_2/\mathrm{pt} \arrow[ul, "D'"] \arrow[dl, "E'"'] & T/\mathrm{pt} \arrow[l, "\ell"'] \arrow[r, "D"] \arrow[u, "G"] \arrow[ul, "D'"'] \arrow[ur, "G'"] \arrow[dl, "E'"] \arrow[d, "E"'] \arrow[dr, "\ell'"'] & Y'_2/\mathrm{pt} \arrow[ur, "G'"'] \arrow[dr, "\ell'"] & \\
X'_6/\mathrm{pt} \arrow[dr, dashed, no head] & Y_1/\mathrm{pt} \arrow[l, "\ell"] \arrow[d, "E"'] & Y'_1 \arrow[dl, "E'"] \arrow[dr, "\ell'"'] & Y''_1/\mathrm{pt} \arrow[r, "D"] \arrow[d, "E"] & X_6/\mathrm{pt} \arrow[dl, dashed, no head] \\
 & \mathcal{Q}/\mathrm{pt} \arrow[rr, dashed, no head] & & \mathcal{Q}/\mathrm{pt} &
\end{tikzcd}$$

$D'_{i_1} := 3F - 2E_1 - 2E_2 - 2E_3 - 2E'_{i_1} - E'_{i_2} - \cdots - E'_{i_4}$ (1)

$G_{i_1} := 4F - 3E_{i_1} - 2E_{i_2} - 2E_{i_3} - 2E'_1 - \cdots - 2E'_4$ (1)

$G'_i := 4F - 3E'_{i_1} - 2E_{i_2} - \cdots - 2E_{i_4} - 2E_1 - 2E_2 - 2E_3$ (1)

We have

$$E_i \cdot E'_j = E_i \cdot \ell'_j = E'_i \cdot \ell = \ell \cdot D'_i = \ell'_i \cdot D = D'_i \cdot G_j = D \cdot G'_i = G_i \cdot G'_j = 0, \quad D'_i \cdot G'_j = 1 - \delta_{ij}$$

for all i, j. All other pairs of orbits have no trivial intersections. Completing the commutative diagram yields the diagram above, where X_6 and X'_6 are del Pezzo surfaces of degree 6. The link joining two del Pezzo surfaces of degree 6 in the diagram is a Geiser link, so the two surfaces are isomorphic.

Remark A.2.2. Any rational del Pezzo surface X_6 of degree 6 with $\rho(X_6) = 1$ has a rational point: this is clear if $\mathbf{k}$ is infinite and for $\mathbf{k}$ finite it follows from [Wei56, p.557], see also [Man86, Theorem 23.1]. Then there is a link of type II $\chi \colon X_6 \dashrightarrow \mathcal{Q}$ that is not defined at a rational point r and contracts a curve D with three geometric components onto a point of degree 3. See Figure A.2 for the following. The anticanonical divisor $H := -K_{X_6}$ generates $\mathrm{NS}(X_6)$, and H is equivalent the sum of the (-1)-curves of $(X_6)_{\overline{\mathbf{k}}}$. The curve $D = D_1 + D_2 + D_3$ on Figure A.2 is equivalent to H and has a triple point at r. The pullback onto X_6 of general curve F generating $\mathrm{NS}(\mathcal{Q})$ has a double point at r and is equivalent to H. We compute that the (-1)-curves in $T_{\overline{\mathbf{k}}}$ are the following, where by abuse of notation $H \subset T_{\overline{\mathbf{k}}}$ denotes the pullback of a general curve generating $\mathrm{NS}(X_6)$ and where the number in parenthesis indicates the number of geometric components:

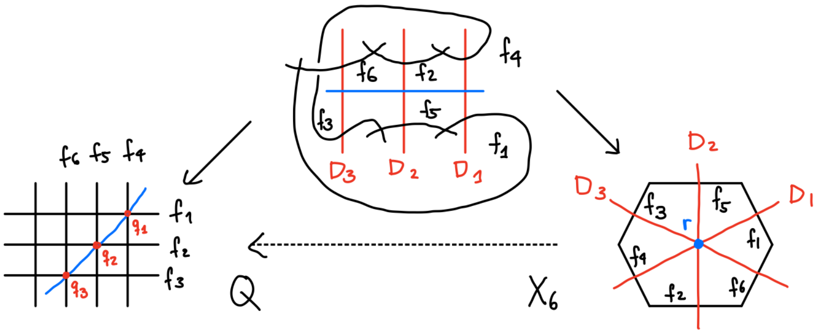


Figure A.2: The link $\chi\colon X_6 \dashrightarrow \mathcal{Q}$

the five exceptional divisors $E_1,\dots,E_5$	
H (6)	$3H-4E_{i_1}-4E_{i_2}-4E_{i_3}-2E_{i_4}-2E_{i_5}$ (2)
$H-3E_i$ (3)	$4H-6E_{i_1}-6E_{i_2}-3E_{i_3}-3E_{i_4}-3E_{i_5}$ (3)
$H-2E_i-2E_j$ (2)	$5H-6E_{i_1}-\cdots-6E_{i_4}$ (6)
$H-2E_{i_1}-E_{i_2}-E_{i_3}-E_{i_4}$ (1)	$5H-3E_{i_1}-6E_{i_2}-\cdots-6E_{i_5}$ (3)
$2H-3E_{i_1}-3E_{i_2}-3E_{i_3}$ (3)	$6H-6E_1-\cdots-6E_5$ (6)
$2H-2E_1-\cdots-2E_5$ (2)	$7H-12E_{i_1}-6E_{i_2}-\cdots-6E_{i_5}$ (6)
$2H-3E_{i_1}-2E_{i_2}-\cdots-2E_{i_5}$ (1)	$11H-12E_1-\cdots-12E_5$ (6)

Since we have already covered all contractions $T\longrightarrow\mathbb{P}^2$ and $T\longrightarrow\mathcal{Q}$, we only need to consider contractions of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves with $\leqslant 5$ members.

The case $(X_6,2,3)$:

Let $T\longrightarrow X_6$ be the blow-up of a point of degree 2 and a point of degree 3 on a del Pezzo surface X_6 of degree 6 with $\rho(X_6)=1$, and denote respectively by E_1,E_2 and E'_1,E'_2,E'_3 the geometric components of their exceptional divisors. From Remark A.2.2 we obtain that the only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves on $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 5$ and whose members are pairwise disjoint are the following, where the number in parenthesis indicates the number of geometric components:

E_1,E_2, $\quad E'_1,E'_2,E'_3$
$A:=H-2E_1-2E_2$ (2)
$D:=2H-3E'_1-3E'_2-E'_3$ (3)
$G_i:=2H-3E_i-2E_{3-i}-2E'_1-\cdots-2E'_3$ (1)
$I_{i_1}:=2H-3E'_{i_1}-2E_1-2E_2-2E'_{i_2}-2E'_{i_3}$ (1)
$J:=3H-2E_1-2E_2-3E'_1-\cdots-3E'_3$ (2)
$N:=4H-6E_1-6E_2-3E'_1-\cdots-3E'_3$ (3)

$$\begin{tikzcd}
 & X_6^5/\mathrm{pt} \arrow[rr, dashed, dash] \arrow[dl, dashed, dash] & & X_6^4/\mathrm{pt} \arrow[dr, dashed, dash] & \\
X_6^6/\mathrm{pt} \arrow[ddd, dashed, dash] & Y_3/\mathrm{pt} \arrow[l, "N"'] \arrow[u, "I"] & Y_3'/\mathrm{pt} \arrow[ul, "G"'] \arrow[ur, "J"] & Y_3''/\mathrm{pt} \arrow[u, "I"'] \arrow[r, "D"] & X_6^3/\mathrm{pt} \arrow[ddd, dashed, dash] \\
 & Y_2/\mathrm{pt} \arrow[ul, "G"] \arrow[ddl, "A"'] & T/\mathrm{pt} \arrow[l, "N"'] \arrow[r, "D"] \arrow[u, "I"] \arrow[ul, "G"'] \arrow[ur, "J"] \arrow[dl, "A"'] \arrow[d, "E'"] \arrow[dr, "E"] & Y_2'/\mathrm{pt} \arrow[ur, "J"'] \arrow[ddr, "E"] & \\
X_6^7/\mathrm{pt} \arrow[dr, dashed, dash] & Y_1/\mathrm{pt} \arrow[l, "N"'] \arrow[d, "E"'] & Y_1'/\mathrm{pt} \arrow[dl, "A"'] \arrow[dr, "E"] & Y_1''/\mathrm{pt} \arrow[r, "D"] \arrow[d, "E'"] & X_6^2/\mathrm{pt} \arrow[dl, dashed, dash] \\
 & X_6^8/\mathrm{pt} \arrow[rr, dashed, dash] & & X_6/\mathrm{pt} &
\end{tikzcd}$$

We have

$$E_i\cdot E'_j=E_i\cdot D=E'_j\cdot A=A\cdot N=D\cdot J=G_i\cdot I_j=G_i\cdot N=I_i\cdot J=0$$

for all i,j. All other pairs of orbits have no trivial intersections. Completing the blow-up diagram with these contractions, we obtain the commutative digram above, where

$X_6^2, \ldots, X_6^8$ are del Pezzo surfaces of degree 6.

The remaining cases:
These are $(\mathcal{Q}, 1, 6)$, $(X_6, 1, 4)$, $(X_5, 1, 3)$ and $(X_5, 2, 2)$, and they already appear in the diagrams of the cases $(\mathbb{P}^2, 2, 6)$, $(\mathcal{Q}, 3, 4)$, $(\mathbb{P}^2, 3, 5)$ and $(\mathcal{Q}, 2, 5)$, respectively.

A.2.2 Elementary relations dominated by a del Pezzo surface of degree 2

Suppose that T/pt is a rank 3 fibration with $K_T^2 = 2$. There is a birational morphism $T_{\overline{\mathbf{k}}} \longrightarrow \mathbb{P}^2_{\overline{\mathbf{k}}}$ blowing up eight points $p_1, \ldots, p_7$ and $T_{\overline{\mathbf{k}}}$ contains precisely 56 (-1)-curves:

- the seven exceptional divisors,
- the 21 strict transform of the lines through two of the p_i,
- the 21 conics passing through five of $p_1, \ldots, p_8$,
- the seven cubics passing through seven of the p_i and singular at one of these seven,

The case $(\mathbb{P}^2, 1, 6)$:
Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a rational point and a point of degree 6, and denote respectively by E be the exceptional divisor of the rational point and $E'_1, \ldots, E'_6$ the geometric components of the exceptional divisors of the point of degree 6. Let $L \subset T$ be the pullback of a general line in $\mathbb{P}^2$. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves on $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 7$ are the following:

$$E, \quad E'_1, \ldots, E'_6,$$
$$\ell_i := L - E - E'_i$$
$$C_{i_6} := 2L - E'_{i_1} - \cdots - E'_{i_5}$$
$$D := 3L - 2E - E'_1 - \cdots - E'_6$$
$$D'_{i_1} := 3L - 2E'_{i_1} - E'_{i_2} - \cdots - E'_{i_6}$$
$$E \cdot E'_i = E \cdot C_i = \ell_i \cdot D = D \cdot D_i = 0$$
$$E'_i \cdot \ell_j = C_i \cdot D'_j = \delta_{ij}, \quad E'_i \cdot C_j = \ell_i \cdot D'_j = 1 - \delta_{ij}$$

$$\begin{tikzcd}
 & \mathbb{P}^2/\mathrm{pt} \arrow[rr, dashed, no head] & & \mathbb{P}^2/\mathrm{pt} & \\
\mathbb{F}_1/\mathbb{P}^1 \arrow[ur, dashed, no head] \arrow[ddd, dashed, no head] & \mathbb{F}_1/\mathrm{pt} \arrow[l] \arrow[u, "D"] & Y_2/\mathrm{pt} \arrow[ul, "D'"'] \arrow[ur, "\ell"] & \mathbb{F}_1/\mathrm{pt} \arrow[r] \arrow[u, "D"] & \mathbb{F}_1/\mathbb{P}^1 \arrow[ul, dashed, no head] \arrow[ddd, dashed, no head] \\
 & T/\mathbb{P}^1 \arrow[ul, "D'"] \arrow[dl, "C"'] & T/\mathrm{pt} \arrow[l] \arrow[r] \arrow[ul, "D'"] \arrow[u, "D"] \arrow[ur, "\ell"] \arrow[dl, "C"'] \arrow[d, "E"] \arrow[dr, "E'"] & T/\mathbb{P}^1 \arrow[ur, "\ell"'] \arrow[dr, "E'"] & \\
 & & & & \\
\mathbb{F}_1/\mathbb{P}^1 \arrow[dr, dashed, no head] & \mathbb{F}_1/\mathrm{pt} \arrow[l] \arrow[d, "E"] & Y_1/\mathrm{pt} \arrow[dl, "C"'] \arrow[dr, "E'"] & \mathbb{F}_1/\mathrm{pt} \arrow[r] \arrow[d, "E"] & \mathbb{F}_1/\mathbb{P}^1 \arrow[dl, dashed, no head] \\
 & \mathbb{P}^2 \arrow[rr, dashed, no head] & & \mathbb{P}^2 &
\end{tikzcd}$$

for all i, j, where δ_{ij} is the Kronecker delta. Completing the contraction diagram, we obtain the commutative diagram above.

The case $(\mathbb{P}^2, 2, 5)$:
Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a point of degree 2 and a point of degree 5, and denote respectively by E_1, E_2 and $E'_1, \ldots, E'_5$ the geometric components of their exceptional divisors. Let $L \subset T$ be the pullback of a general line. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves on T of cardinality $\leqslant 7$ are the following:

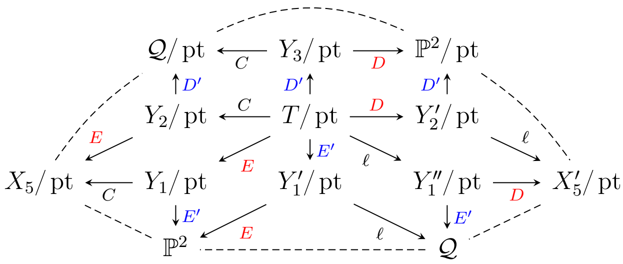
$E_1, E_2, \quad E_1', \dots, E_5'$

$\ell := L - E_1 - E_2$

$C := 2L - E_1' - \dots - E_5'$

$D_i := 3L - 2E_i - E_{3-i} - E_1' - \dots - E_5'$

$D_{i_1}' := 3L - 2E_{i_1}' - E_1 - E_2 - E_{i_2}' - \dots - E_{i_5}'$



We have

$$E_i \cdot E_j' = E_i \cdot C = E_j' \cdot \ell = \ell \cdot D_i = C \cdot D_j' = D_i \cdot D_j' = 0$$

for all i, j. All other pairs of orbits have no trivial intersections. Completing the blow-up diagram, we obtain the commutative diagram above, where X_5 and X_5' are del Pezzo surfaces of degree 5.

The case $(\mathbb{P}^2, 3, 4)$:

Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a point of degree 3 and a point of degree 4, and denote respectively by E_1, E_2, E_3 and $E_1', \dots, E_4'$ the geometric components of their exceptional divisors. Let $L \subset T$ be the pullback of a general line. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves on T of cardinality $\leqslant 7$ and with pairwise disjoint members are the following:

$E_1, E_2, E_3, \quad E_1', \dots, E_4'$

$\ell_{i_3} := L - E_{i_1} - E_{i_2}$

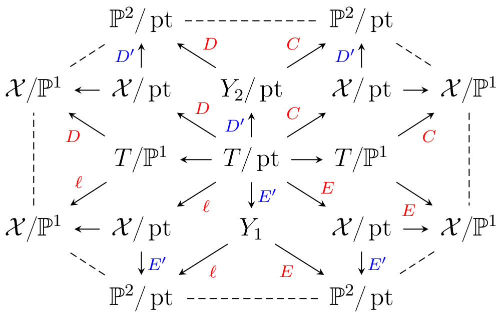
$C_i := 2L - E_i - E_1' - \dots - E_4'$

$D_{i_1} := 3L - 2E_{i_1} - E_{i_2} - E_{i_3} - E_1' - \dots - E_4'$

$D_{i_1}' := 3L - 2E_{i_1}' - E_{i_2}' - \dots - E_{i_4}' - E_1 - E_2$

$E_i \cdot E_j' = E_j' \cdot \ell_i = C_i \cdot D_j' = D_i \cdot D_j' = 0$

$E_i \cdot C_j = \ell_i \cdot D_j = \delta_{ij}, \quad C_i \cdot D_j = 1 - \delta_{ij}$



for all $i, j = 1, 2, 3$. All other pairs of orbits have no trivial intersections. Completing the blow-up diagram, we obtain the commutative diagram above.

Remark A.2.3. If $T_{\overline{\mathbf{k}}} \longrightarrow \mathcal{Q}_{\overline{\mathbf{k}}}$ is a blow-up, then from Remark A.2.1 we deduce that the 56 (-1)-curves on $T_{\overline{\mathbf{k}}}$ are as follows, where F is the curve generating $\mathrm{NS}(\mathcal{Q})$ and the number of geometric components is indicated in parenthesis:

- six exceptional divisors E_i (1)
- $F - 2E_i$ (2)
- $F - E_{i_1} - E_{i_2} - E_{i_3}$ (1)
- $3F - 2E_{i_1} - \dots - 2E_{i_5}$ (2)
- $2F - 2E_{i_1} - E_{i_2} - \dots - E_{i_6}$ (1)

The case $(\mathcal{Q}, 2, 4)$:

Let $T \longrightarrow \mathcal{Q}$ be the blow-up of a point of degree 2 and a point of degree 4, and denote respectively by E_1, E_2 and $E_1', \dots, E_4'$ the geometric components of their exceptional divisors. Let $F \subset T$ be the pullback of the curve generating $\mathrm{NS}(\mathcal{Q})$. From Remark A.2.3 we obtain that the only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 6$ and whose members are pairwise disjoint are the following, where the number in parentheses indicates the number of geometric components:

$E_1, E_2, \quad E'_1, \ldots, E'_4$

$\ell_i := F - E_1 - E_2 - E'_i$ (1)

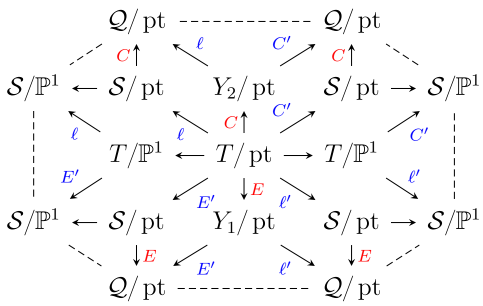
$\ell'_{i_4} := F - E'_{i_1} - E'_{i_2} - E'_{i_3}$ (1)

$C_i := 2F - 2E_i - E_{3-i} - E'_1 - \cdots - E'_4$ (1)

$C'_{i_1} := 2F - 2E'_{i_1} - E_1 - E_2 - E'_{i_2} - \cdots - E'_{i_4}$ (1)

$E_i \cdot E'_j = E_i \cdot \ell'_j = \ell_i \cdot C_j = C_i \cdot C'_j = 0$

$E'_i \cdot \ell_j = \ell_i \cdot C'_j = \ell'_i \cdot C'_j = \delta_{ij}, \quad E'_i \cdot \ell'_j = 1 - \delta_{ij}$



for all i, j. All other pairs of orbits have no trivial intersections. Completing the blow-up diagram, we obtain the commutative diagram above.

<u>The case $(\mathcal{Q}, 3, 3)$:</u>
Let $T \longrightarrow \mathcal{Q}$ be the blow-up of two points of degree 3 and denote respectively by E_1, E_2, E_3 and E'_1, E'_2, E'_3 the geometric components of their exceptional divisors. Let $F \subset T$ be the pullback of the curve generating $\mathrm{NS}(\mathcal{Q})$. From Remark A.2.3 we obtain that the only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 6$ and whose members are pairwise disjoint are the following, where the number in paretheses indicates the number of geometric components:

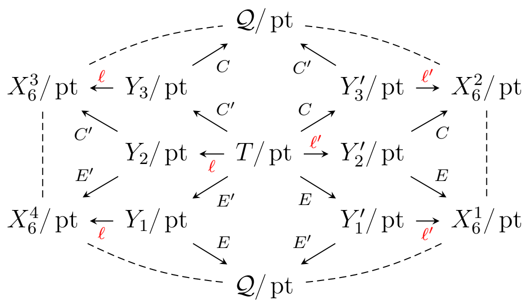
$E_1, E_2, E_3, \quad E'_1, E'_2, E'_3$

$\ell := F - E_1 - E_2 - E_3$ (1)

$\ell' := F - E'_1 - E'_2 - E'_3$ (1)

$C_{i_1} := 2F - 2E_{i_1} - E_{i_2} - E_{i_3} - E'_1 - E'_2 - E'_3$ (3)

$C'_{i_1} := 2F - 2E'_{i_1} - E'_{i_2} - E'_{i_3} - E_1 - E_2 - E_3$ (3)



We have

$$E_i \cdot E'_j = E_i \cdot \ell' = E'_j \cdot \ell = \ell \cdot C'_j = \ell' \cdot C_i = C_i \cdot C'_j = 0, \quad i, j = 1, \ldots, 3.$$

All other pairs of orbits have no trivial intersections. Completing the blow-up diagram, we obtain the commutative diagram above, where $X_6^1, \ldots, X_6^4$ are del Pezzo surfaces of degree 6.

Remark A.2.4. Let $T \longrightarrow X_6$ be a birational morphism to a del Pezzo surface of degree 6 with $\rho(X_6) = 1$. By Remark A.2.2, the curves on T of negative self-intersection are as follows, where H is the pullback of the curve generating $\mathrm{NS}(X_6)$ and the number of geometric components is indicated in parenthesis:

four exceptional divisors E_i (1)
H (6)
$H-3E_i$ (3)
$H-2E_i-2E_j$ (2)
$H-2E_{i_1}-E_{i_2}-E_{i_3}-E_{i_4}$ (1)
$2H-3E_{i_1}-3E_{i_2}-3E_{i_3}$ (3)
$5H-6E_1-\cdots-6E_4$ (6)

Since we have already covered the birational contractions $T\longrightarrow\mathbb{P}^2$ and $T\longrightarrow\mathcal{Q}$, we only need to consider $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves with $\leqslant 4$ members.

The case $(X_6,2,2)$:
Let $T\longrightarrow X_6$ be the blow-up of two points of degree 2 on a del Pezzo surface X_6 of degree 6 with $\rho(X_6)=1$, and denote respectively by E_1,E_2 and E_1',E_2' the geometric components of their exceptional divisors. Let $H=-K_{X_6}$ be the pullback of the curve generating $\mathrm{NS}(X_6)$. From Remark A.2.4, we obtain that the only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 4$ are the following, where the number in parentheses indicates the number of geometric components:

$E_1,E_2,\quad E_1',E_2'$
$\ell_{i_1}:=H-2E_{i_1}-E_{i_2}-E_1'-E_2'$ (1)
$\ell_{i_1}':=H-2E_{i_1}'-E_{i_2}'-E_1-E_2$ (1)
$C:=H-2E_1-2E_2$ (2)
$C':=H-2E_1'-2E_2'$ (2)

$$\begin{tikzcd}
 & & X_6^4/\mathrm{pt} \arrow[dll, dash, dashed] \arrow[drr, dash, dashed] & & \\
X_6^5/\mathrm{pt} \arrow[dd, dash, dashed] & Y_3/\mathrm{pt} \arrow[l, "C"'] \arrow[ur, "\ell'"] & & Y_3'/\mathrm{pt} \arrow[ul, "\ell"'] \arrow[r, "C'"] & X_6^3/\mathrm{pt} \arrow[dd, dash, dashed] \\
 & Y_2/\mathrm{pt} \arrow[ul, "\ell"] \arrow[dl, "E'"'] & T/\mathrm{pt} \arrow[l, "C"] \arrow[r, "C'"] \arrow[ul, "\ell"] \arrow[ur, "\ell'"] \arrow[dl, "E'"] \arrow[dr, "E"'] & Y_2'/\mathrm{pt} \arrow[ur, "\ell'"'] \arrow[dr, "E"] & \\
X_6^6/\mathrm{pt} \arrow[drr, dash, dashed] & Y_1/\mathrm{pt} \arrow[l, "C"] \arrow[dr, "E"] & & Y_1'/\mathrm{pt} \arrow[r, "C'"'] \arrow[dl, "E'"'] & X_6^2/\mathrm{pt} \arrow[dll, dash, dashed] \\
 & & X_6/\mathrm{pt} & &
\end{tikzcd}$$

We have

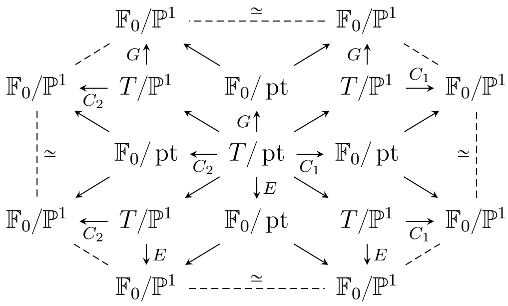
$$E_i\cdot E_j'=E_i\cdot C'=E_j'\cdot C=\ell_i\cdot\ell_j'=\ell_i\cdot C=\ell_i'\cdot C'=0,\quad i,j=1,2.$$

All other pairs of orbits have no trivial intersections. Completing the blow-up diagram, we obtain the commutative diagram above, where $X_6^2,\dots,X_6^6$ are del Pezzo surfaces of degree 6.

The remaining cases:
These are $(\mathcal{Q},1,5)$, $(X_6,1,3)$ and $(X_5,1,2)$, and they appear in the diagrams of the cases $(\mathbb{P}^2,2,5)$, $(\mathcal{Q},3,3)$ and $(\mathbb{P}^2,2,5)$, respectively.

The case $(\mathbb{F}_0,6)$:
Let $T\longrightarrow\mathbb{F}_0$ be the blow-up of a point of degree 6, and denote by $E_1,\dots,E_6$ the geometric components of its exceptional divisor. Let F_1,F_2 the the classes of the fibres of the two fibrations of $\mathbb{F}_0$. From Remark A.2.3 we obtain that the only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 7$ are the following, where the number in parentheses indicates the number of geometric components:

$E_1, \dots, E_6$
$C_{ji} := F_j - E_i$
$D_{1i_6} := F_1 + 2F_2 - E_{i_1} - \cdots - E_{i_5}$
$D_{2i_6} := 2F_1 + F_2 - E_{i_1} - \cdots - E_{i_5}$
$G_{i_1} := 2(F_1 + F_2) - 2E_{i_1} - E_{i_2} - \cdots - E_{i_6}$



We have

$$E_iC_{1j} = E_iC_{2j} = C_{1i}D_{1j} = D_{1i}G_j = D_{2i}G_j = \delta_{ij}, \quad C_{1i}G_j = C_{2i}G_j = C_{1i}C_{2j} = 1 - \delta_{ij},$$

and $C_{1i}D_{2j} = C_{2i}D_{1j} = \delta_{ij} + 1$. Completing the blow-up diagram, we obtain the commutative diagram above.

A.2.3 Elementary relations dominated by a del Pezzo surface of degree 3

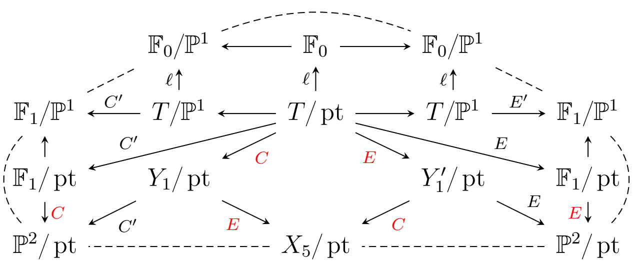
Suppose that $T/\,\mathrm{pt}$ is a rank 3 fibration with $K_T^2 = 3$. There is a birational morphism $T_{\overline{\mathbf{k}}} \longrightarrow \mathbb{P}^2_{\overline{\mathbf{k}}}$ blowing up eight points $p_1, \dots, p_6$ and $T_{\overline{\mathbf{k}}}$ contains precisely 27 (-1)-curves:

- the six exceptional divisors,
- the 15 strict transform of the lines through two of the p_i,
- the 6 conics passing through five of $p_1, \dots, p_6$.

<u>The case $(\mathbb{P}^2, 1, 5)$:</u>

Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a rational point and a point of degree 5, and denote respectively by E the exceptional divisor of the rational point and $E'_1, \dots, E'_5$ the geometric components of the exceptional divisors of the point of degree 5. Let $L \subset T$ be the pullback of a general line in $\mathbb{P}^2$. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves on $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 6$ are:

E, $\quad E'_1, \dots, E'_5$
$\ell_i := L - E - E'_i$
$C'_{i_5} := 2L - E - E'_{i_1} - \cdots - E'_{i_4}$
$C := 2L - E'_1 - \cdots - E'_5$
$E \cdot E'_j = E \cdot C = C'_j \cdot C = 0$
$E'_i \cdot \ell_j = \ell_i \cdot C'_j = \delta_{ij}, \quad E'_i \cdot C'_j = 1 - \delta_{ij}$



for all i, j. All other pairs of orbits have no trivial intersections. This yields all possible contractions from T/B to a rank 2 fibration. Completing the contraction diagram starting from T, we obtain the commutative diagram above, where X_5 is a del Pezzo surface of degree 5.

<u>The case $(\mathbb{P}^2, 2, 4)$:</u>

Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a point of degree 2 and a point of degree 4, and denote respectively by E_1, E_2 and $E_1', \ldots, E_4'$ the geometric components of their exceptional divisors. Let $L \subset T$ be the pullback of a general line in $\mathbb{P}^2$. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 6$ and pairwise disjoint members are:

$$
\begin{aligned}
& E_1, E_2, \quad E_1', \ldots, E_4' \\
& \ell := L - E_1 - E_2, \\
& C_i := 2L - E_i - E_1' - \cdots - E_4' \\
& C_{i_4}' := 2L - E_1 - E_2 - E_{i_1}' - \cdots - E_{i_3}' \\
& E_i \cdot E_j' = E_j' \cdot \ell = \ell \cdot C_i' = C_i' \cdot C_j = 0 \\
& E_i \cdot C_j = \delta_{ij}, \quad E_i' \cdot C_j' = 1 - \delta_{ij}
\end{aligned}
$$

$$
\begin{tikzcd}
\mathcal{X}/\mathbb{P}^1 \arrow[dd, dashed, no head] \arrow[rrrr, dashed, bend left, no head] & & T/\mathbb{P}^1 \arrow[ll, "E"'] \arrow[rr, "C"] & & \mathcal{X}/\mathbb{P}^1 \arrow[dd, dashed, no head] \\
& \mathcal{X}/\mathrm{pt} \arrow[ul] \arrow[dl, "E'"'] & T/\mathrm{pt} \arrow[u] \arrow[l, "E"'] \arrow[r, "C"] \arrow[dl, "E'"'] \arrow[d, "\ell"] \arrow[dr, "C'"] & \mathcal{X}/\mathrm{pt} \arrow[ur] \arrow[dr, "C'"] & \\
\mathbb{P}^2/\mathrm{pt} \arrow[dr, dashed, no head] & Y_1/\mathrm{pt} \arrow[l, "E"] \arrow[d, "\ell"] & Y_1'/\mathrm{pt} \arrow[dl, "E'"] \arrow[dr, "C'"] & Y_1''/\mathrm{pt} \arrow[r, "C"] \arrow[d, "\ell"] & \mathbb{P}^2/\mathrm{pt} \arrow[dl, dashed, no head] \\
& \mathcal{Q}/\mathrm{pt} \arrow[rr, dashed, no head] & & \mathcal{Q}/\mathrm{pt} &
\end{tikzcd}
$$

for all i, j. All other pairs of orbits have no trivial intersections. Completing the contraction diagram starting from T, we obtain the commutative diagram above.

<u>The case $(\mathbb{P}^2, 3, 3)$:</u>

Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of two points of degree 3 and denote respectively by E_1, E_2, E_3 and E_1', E_2', E_3' the geometric components of their exceptional divisors. Let $L \subset T$ be the pullback of a general line in $\mathbb{P}^2$. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 6$ and pairwise disjoint members are:

$$
\begin{aligned}
& E_1, E_2, E_3, \quad E_1', E_2', E_3' \\
& \ell_{i_3} := L - E_{i_1} - E_{i_2} \\
& \ell_{i_3}' := L - E_{i_1}' - E_{i_2}' \\
& C_{i_3} := 2L - E_1 - E_2 - E_3 - E_{i_1}' - E_{i_2}' \\
& C_{i_3}' := 2L - E_{i_1} - E_{i_2} - E_1' - E_2' - E_3'
\end{aligned}
$$

$$
\begin{tikzcd}
& & \mathbb{P}^2/\mathrm{pt} \arrow[drr, dashed, no head] & & \\
\mathbb{P}^2/\mathrm{pt} \arrow[urr, dashed, no head] \arrow[dd, dashed, no head] & Y_3/\mathrm{pt} \arrow[l, "\ell"] \arrow[ur, "C'"] & & Y_3'/\mathrm{pt} \arrow[ul, "C"'] \arrow[r, "\ell'"] & \mathbb{P}^2/\mathrm{pt} \arrow[dd, dashed, no head] \\
& Y_2/\mathrm{pt} \arrow[ul, "C"] \arrow[dl, "E'"'] & T/\mathrm{pt} \arrow[l, "\ell"] \arrow[r, "\ell'"] \arrow[ul, "C"'] \arrow[ur, "C'"] \arrow[dl, "E'"] \arrow[dr, "E"'] & Y_2'/\mathrm{pt} \arrow[ur, "C'"'] \arrow[dr, "E"] & \\
\mathbb{P}^2/\mathrm{pt} \arrow[drr, dashed, no head] & Y_1/\mathrm{pt} \arrow[l, "\ell"] \arrow[dr, "E"] & & Y_1'/\mathrm{pt} \arrow[r, "\ell'"] \arrow[dl, "E'"'] & \mathbb{P}^2/\mathrm{pt} \\
& & \mathbb{P}^2/\mathrm{pt} \arrow[urr, dashed, no head] & &
\end{tikzcd}
$$

We have

$$
\begin{aligned}
& E_i \cdot E_j' = E_i \cdot \ell_j' = E_j' \cdot \ell_i = \ell_i \cdot C_j = \ell_i' \cdot C_j' = C_i \cdot C_j' = 0, \\
& E_i \cdot C_j' = E_i' \cdot C_j = 1 - \delta_{ij}, \quad \ell_i \cdot C_j' = \ell_i' \cdot C_i = \delta_{ij}
\end{aligned}
$$

for $i, j = 1, 2, 3$. All other pairs of orbits have no trivial intersections. Completing the contraction diagram starting from T, we obtain the diagram above.

<u>The case $(\mathcal{Q}, 2, 3)$:</u>

Let $T \longrightarrow \mathcal{Q}$ be the blow-up of a point of degree 2 and a point of degree 3, and denote respectively by E_1, E_2 and E_1', E_2', E_3' the geometric components of their exceptional divisors. Let $F \subset T$ be the pullback of the curve generating $\mathrm{NS}(\mathcal{Q})$. From Remark A.2.1 we obtain that the only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 5$ and pairwise disjoint members are:

$E_1, E_2, \quad E_1', E_2', E_3'$
$\ell := F - E_1' - E_2' - E_3'$ (1)
$D_i := F - E_1 - E_2 - E_i'$ (1)
$C := 3F - 2E_1 - 2E_2 - 2E_1' - 2E_2' - 2E_3'$ (2)
$E_i \cdot E_j' = E_i \cdot \ell = \ell \cdot C = C \cdot D_i = 0$
$E_i' \cdot D_j = \delta_{ij}$

$$\begin{tikzcd}
\mathcal{S}/\mathbb{P}^1 \arrow[dd, dash, dashed] \arrow[rrrr, dash, dashed, bend left] & & T/\mathbb{P}^1 \arrow[ll, "E'"'] \arrow[rr, "D"] & & \mathcal{S}/\mathbb{P}^1 \arrow[dd, dash, dashed] \\
& \mathcal{S}/\mathrm{pt} \arrow[ul] \arrow[dl, "E"'] & T/\mathrm{pt} \arrow[l, "E'"'] \arrow[r, "D"] \arrow[u] \arrow[dl, "E"'] \arrow[d, "\ell"] \arrow[dr, "C"] & \mathcal{S}/\mathrm{pt} \arrow[ur] \arrow[dr, "C"] & \\
\mathcal{Q}/\mathrm{pt} \arrow[dr, dash, dashed] & Y_1/\mathrm{pt} \arrow[l, "E'"] \arrow[d, "\ell"] & Y_1'/\mathrm{pt} \arrow[dl, "E"'] \arrow[dr, "C"] & Y_1''/\mathrm{pt} \arrow[r, "D"] \arrow[d, "\ell"] & \mathcal{Q}/\mathrm{pt} \arrow[dl, dash, dashed] \\
& X_6/\mathrm{pt} \arrow[rr, dash, dashed] & & X_6'/\mathrm{pt} &
\end{tikzcd}$$

for all i, j. All other pairs of orbits have no trivial intersections. Completing the contraction diagram starting from T, we obtain the diagram above, where X_6 and X_6' are del Pezzo surfaces of degree 6.

The remaining cases:
They are $(\mathcal{Q}, 1, 4)$, $(X_6, 1, 2)$ and $(X_5, 1, 1)$, and they appear in the diagrams of the cases $(\mathbb{P}^2, 2, 4)$, $(\mathcal{Q}, 2, 3)$ and $(\mathbb{P}^2, 1, 5)$, respectively.

A.2.4 Elementary relations dominated by a del Pezzo surface of degree 4

Suppose that T/pt is a rank 3 fibration with $K_T^2 = 4$. There is a birational morphism $T_{\overline{\mathbf{k}}} \longrightarrow \mathbb{P}^2_{\overline{\mathbf{k}}}$ blowing up eight points $p_1, \dots, p_5$ and $T_{\overline{\mathbf{k}}}$ contains precisely 16 (-1)-curves: the five exceptional divisors, the 10 strict transform of the lines through two of the p_i, and the conic passing through five of $p_1, \dots, p_5$.

The case $(\mathbb{P}^2, 1, 4)$:
Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a rational point and a point of degree 4, and denote by E the exceptional divisor of the rational point and by $E_1', \dots, E_4'$ the geometric components of their exceptional divisors of the point of degree 4. Let $L \subset T$ be the pullback of a general line. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 5$ are:

$E, \quad E_1', \dots, E_4'$
$\ell_i := L - E - E_i'$
$C := 2L - E - E_1' - \cdots - E_4'$
$E \cdot E_j' = C \cdot \ell_j = 0, \quad E_i' \cdot \ell_j = \delta_{ij}$

$$\begin{tikzcd}
& & \mathbb{P}^2/\mathrm{pt} \arrow[drr, dash, dashed] & & \\
\mathcal{X}/\mathbb{P}^1 \arrow[urr, dash, dashed] \arrow[dd, dash, dashed] & \mathcal{X}/\mathrm{pt} \arrow[l] \arrow[ur, "\ell"] & & \mathbb{F}_1/\mathrm{pt} \arrow[r] \arrow[ul, "C"'] & \mathbb{F}_1/\mathbb{P}^1 \arrow[dd, dash, dashed] \\
& T/\mathbb{P}^1 \arrow[ul, "C"] \arrow[dl, "E"'] & T/\mathrm{pt} \arrow[l] \arrow[r] \arrow[ul, "C"] \arrow[ur, "\ell"'] \arrow[dl, "E"'] \arrow[dr, "E'"] & T/\mathbb{P}^1 \arrow[ur, "\ell"'] \arrow[dr, "E'"] & \\
\mathcal{X}/\mathbb{P}^1 \arrow[drr, dash, dashed] & \mathcal{X}/\mathrm{pt} \arrow[l] \arrow[dr, "E'"] & & \mathbb{F}_1/\mathrm{pt} \arrow[r] \arrow[dl, "E"'] & \mathbb{F}_1/\mathbb{P}^1 \\
& & \mathbb{P}^2/\mathrm{pt} \arrow[urr, dash, dashed] & &
\end{tikzcd}$$

All other pairs of orbits have no trivial intersections. By completing the contraction diagram starting from T, we obtain the above commutative diagram.

The case $(\mathbb{P}^2, 2, 3)$:
Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a point of degree 2 and a point of degree 3, and denote respectively by E_1, E_2 and E_1', E_2', E_3' the geometric components of their exceptional divisors. Let $L \subset T$ be the pullback of a general line. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves

in $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 5$ are:

$E_1, E_2, \quad E_1', E_2', E_3'$
$\ell := L - E_1 - E_2$
$\ell'_{i_3} := L - E'_{i_1} - E'_{i_2}$
$C := 2L - E_1 - E_2 - E_1' - E_2' - E_3'$
$E_i \cdot E_j' = E_i \cdot \ell_j' = E_j' \cdot \ell = \ell \cdot C = \ell_j' \cdot C = 0$

$$\begin{tikzcd}
 & & X_6/\mathrm{pt} \arrow[dll, dashed, no head] \arrow[drr, dashed, no head] & & \\
 & Y_2/\mathrm{pt} \arrow[ur, "\ell"] \arrow[dl, "\ell'"'] & T/\mathrm{pt} \arrow[l, "C"'] \arrow[r, "\ell"] \arrow[d, "E"] \arrow[dl, "\ell'"] \arrow[dr, "E'"] & Y_2'/\mathrm{pt} \arrow[ul, "C"'] \arrow[dr, "E'"] & \\
\mathcal{Q}/\mathrm{pt} \arrow[dr, dashed, no head] & Y_1/\mathrm{pt} \arrow[l, "C"] \arrow[d, "E"] & Y_1'/\mathrm{pt} \arrow[dl, "\ell'"] \arrow[dr, "E'"] & Y_1''/\mathrm{pt} \arrow[r, "\ell"'] \arrow[d, "E"] & \mathcal{Q}/\mathrm{pt} \arrow[dl, dashed, no head] \\
 & \mathbb{P}^2/\mathrm{pt} \arrow[rr, dashed, no head] & & \mathbb{P}^2/\mathrm{pt} &
\end{tikzcd}$$

for $i = 1,2$, $j = 1,2,3$. All other pairs of orbits have no trivial intersections. By completing the contraction diagram starting from T, we obtain the above commutative diagram, where X_6 is a del Pezzo surface of degree 6.

The case $(\mathcal{Q},2,2)$:
Let $T \longrightarrow \mathcal{Q}$ be the blow-up of two points of degree 2 and denote respectively by E_1, E_2 and E_1', E_2' the geometric components of their exceptional divisors. Let $F \subset T$ be the pullback of a general curve generating $\mathrm{NS}(\mathcal{Q})$. From Remark A.2.1 we obtain that the only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ with pairwise disjoint members are:

$E_1, E_2, \quad E_1', E_2'$
$\ell_i := F - E_i - E_1' - E_2'$
$\ell_i' := F - E_1 - E_2 - E_i'$
$E_i \cdot E_j' = \ell_i \cdot \ell_j' = 0, \quad i,j = 1,2$
$E_i \cdot \ell_j = E_i' \cdot \ell_j' = \delta_{ij}, \quad i,j = 1,2$

$$\begin{tikzcd}
 & & \mathcal{Q}/\mathrm{pt} \arrow[dll, dashed, no head] \arrow[drr, dashed, no head] & & \\
\mathcal{S}/\mathbb{P}^1 \arrow[dd, dashed, no head, bend right] & \mathcal{S}/\mathrm{pt} \arrow[l] \arrow[ur, "\ell'"] & & \mathcal{S}/\mathrm{pt} \arrow[r] \arrow[ul, "\ell"'] & \mathcal{S}/\mathbb{P}^1 \arrow[dd, dashed, no head, bend left] \\
T/\mathbb{P}^1 \arrow[u, "\ell"] \arrow[d, "E"] & & T/\mathrm{pt} \arrow[ll] \arrow[rr] \arrow[ul, "\ell"] \arrow[ur, "\ell'"'] \arrow[dl, "E"] \arrow[dr, "E'"'] & & T/\mathbb{P}^1 \arrow[u, "\ell'"'] \arrow[d, "E'"'] \\
\mathcal{S}/\mathbb{P}^1 & \mathcal{S}/\mathrm{pt} \arrow[l] \arrow[dr, "E'"'] & & \mathcal{S}/\mathrm{pt} \arrow[r] \arrow[dl, "E"] & \mathcal{S}/\mathbb{P}^1 \\
 & & \mathcal{Q}/\mathrm{pt} \arrow[ull, dashed, no head] \arrow[urr, dashed, no head] & &
\end{tikzcd}$$

By completing the contraction diagram starting from T, we obtain the above diagram.

The remaining cases:
They are $(\mathcal{Q},3,1)$ and $(X_6,1,1)$, which both appear in the diagram of $(\mathbb{P}^2,2,3)$.

The case $(\mathbb{F}_0,4)$:
Let $T \longrightarrow \mathbb{F}_0$ be the blow-up of a point of degree 4, and denote by $E_1,\dots,E_4$ the geometric components of its exceptional divisor. Let F_1, F_2 the the classes of the fibres of the two fibrations of $\mathbb{F}_0$. From Remark A.2.3 we obtain that the only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 5$ are the following, where the number in parentheses indicates the number of geometric components:

$E_1, \dots, E_4$
$C_{ji} := F_j - E_i$
$D_{i_4} := F_1 + F_2 - E_{i_1} - E_{i_2} - E_{i_3}$
$E_i C_{1j} = E_i C_{2j} = C_{1i} D_{1j} = \delta_{ij}$
$C_{1i} C_{2j} = 1 - \delta_{ij}$

$$\begin{tikzcd}
 & \mathbb{F}_0/\mathbb{P}^1 \arrow[rr, dashed, no head, "\simeq"] \arrow[dl, dashed, no head] & & \mathbb{F}_0/\mathbb{P}^1 \arrow[dr, dashed, no head] & \\
\mathbb{F}_0/\mathbb{P}^1 \arrow[dd, dashed, no head, "\simeq"] & T/\mathbb{P}^1 \arrow[l, "C_2"] \arrow[u, "G"] & \mathbb{F}_0/\mathrm{pt} \arrow[ul] \arrow[ur] & T/\mathbb{P}^1 \arrow[r, "C_1"] \arrow[u, "G"] & \mathbb{F}_0/\mathbb{P}^1 \arrow[dd, dashed, no head, "\simeq"'] \\
 & \mathbb{F}_0/\mathrm{pt} \arrow[ul] \arrow[dl] & T/\mathrm{pt} \arrow[l, "C_2"] \arrow[r, "C_1"'] \arrow[u, "G"] \arrow[d, "E"] \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \mathbb{F}_0/\mathrm{pt} \arrow[ur] \arrow[dr] & \\
\mathbb{F}_0/\mathbb{P}^1 \arrow[dr, dashed, no head] & T/\mathbb{P}^1 \arrow[l, "C_2"] \arrow[d, "E"] & \mathbb{F}_0/\mathrm{pt} \arrow[dl] \arrow[dr] & T/\mathbb{P}^1 \arrow[r, "C_1"'] \arrow[d, "E"] & \mathbb{F}_0/\mathbb{P}^1 \arrow[dl, dashed, no head] \\
 & \mathbb{F}_0/\mathbb{P}^1 \arrow[rr, dashed, no head, "\simeq"] & & \mathbb{F}_0/\mathbb{P}^1 &
\end{tikzcd}$$

Completing the blow-up diagram, we obtain the commutative diagram above.

A.2.5 Elementary relations dominated by a del Pezzo surface of degree 5

Suppose that T/pt is a rank 3 fibration with $K_T^2 = 5$. There is a birational morphism $T_{\overline{\mathbf{k}}} \longrightarrow \mathbb{P}^2_{\overline{\mathbf{k}}}$ blowing up eight points $p_1, \ldots, p_4$ and $T_{\overline{\mathbf{k}}}$ contains precisely ten (-1)-curves: the four exceptional divisors and the six strict transform of the lines through two of the p_i.

The case $(\mathbb{P}^2, 1, 3)$: Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a rational point and a point of degree 3, and denote by E the exceptional divisor of the rational point and by E'_1, E'_2, E'_3 the geometric components of their exceptional divisor of the point of degree 3. Let $L \subset T$ be the pullback of a general line. The $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ are:

$$
\begin{aligned}
&E, \quad E'_1, \ldots, E'_3\\
&\ell_i := L - E - E'_i\\
&\ell'_{i_3} := L - E'_{i_1} - E'_{i_2}\\
&E \cdot E'_i = E \cdot \ell'_i = 0\\
&\ell_i \cdot \ell'_j = E'_i \cdot \ell_j = \delta_{ij}
\end{aligned}
$$

$$
\begin{tikzcd}
\mathbb{F}_0/\mathbb{P}^1 \arrow[dddd, dashed, no head] \arrow[rrrr, dashed, bend left, no head] & & \mathbb{F}_0/\mathrm{pt} \arrow[ll] \arrow[rr] & & \mathbb{F}_0/\mathbb{P}^1 \arrow[dddd, dashed, no head] \\
& T/\mathbb{P}^1 \arrow[ul, "\ell"] \arrow[dl, "\ell'"] & T/\mathrm{pt} \arrow[l] \arrow[r] \arrow[u, "\ell"] \arrow[dl, "\ell'"] \arrow[d, "E"] \arrow[dr, "E'"] & T/\mathbb{P}^1 \arrow[ur, "\ell"'] \arrow[dr, "E'"'] & \\
& \mathbb{F}_1/\mathrm{pt} \arrow[d, "E"] \arrow[dl] & Y_1/\mathrm{pt} \arrow[dl, "\ell'"] \arrow[dr, "E'"'] & \mathbb{F}_1/\mathrm{pt} \arrow[d, "E"] \arrow[dr] & \\
\mathbb{F}_1/\mathbb{P}^1 \arrow[r, dashed, no head] & \mathbb{P}^1/\mathrm{pt} \arrow[rr, dashed, no head] & & \mathbb{P}^2/\mathrm{pt} & \mathbb{F}_1/\mathbb{P}^1 \arrow[l, dashed, no head]
\end{tikzcd}
$$

for all i, j. By completing the contraction diagram starting from T, we obtain the above diagram.

The case $(\mathbb{P}^2, 2, 2)$:
Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of two points of degree 2 and denote respectively by E_1, E_2 and E'_1, E'_2 the geometric components of their exceptional divisors. Let $L \subset T$ be the pullback of a general line. The only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ with pairwise disjoint members are:

$$
\begin{aligned}
&E_1, E_2, \quad E'_1, E'_2\\
&\ell := L - E_1 - E_2, \quad \ell' := L - E'_1 - E'_2\\
&\ell \cdot \ell' = 1, \quad E_i \cdot \ell_j = E'_i \cdot \ell'_j = 2\\
&E_i \cdot E'_j = E_i \cdot \ell' = E'_i \cdot \ell = 0
\end{aligned}
$$

$$
\begin{tikzcd}
\mathcal{S}/\mathbb{P}^1 \arrow[rrrr, dashed, bend left, no head] \arrow[dd, dashed, bend right, no head] & & T/\mathbb{P}^1 \arrow[ll, "\ell'"'] \arrow[rr, "\ell"] & & \mathcal{S}/\mathbb{P}^1 \arrow[dd, dashed, bend left, no head] \\
\mathcal{S}/\mathrm{pt} \arrow[u] \arrow[d, "E"] & & T/\mathrm{pt} \arrow[u] \arrow[ll, "\ell'"'] \arrow[rr, "\ell"] \arrow[dl, "E"'] \arrow[dr, "E'"] & & \mathcal{S}/\mathrm{pt} \arrow[u] \arrow[d, "E'"'] \\
\mathcal{Q}/\mathrm{pt} \arrow[dr, dashed, bend right, no head] & Y_7/\mathrm{pt} \arrow[l, "\ell'"'] \arrow[dr, "E'"'] & & Y_7/\mathrm{pt} \arrow[r, "\ell"] \arrow[dl, "E"] & \mathcal{Q}/\mathrm{pt} \arrow[dl, dashed, bend left, no head] \\
& & \mathbb{P}^2/\mathrm{pt} & &
\end{tikzcd}
$$

For all i, j. By completing the contraction diagram starting from T, we obtain the above diagram.

The remaining case $(\mathcal{Q}, 1, 2)$ appears in $(\mathbb{P}^2, 2, 2)$.

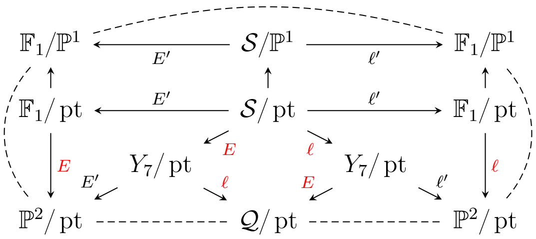
A.2.6 Elementary relations dominated by a del Pezzo surface of degree 6 or degree 7

Suppose that T/pt is a rank 3 fibration with $K_T^2 = 6$. There is a birational morphism $T_{\overline{\mathbf{k}}} \longrightarrow \mathbb{P}^2_{\overline{\mathbf{k}}}$ blowing up eight points $p_1, \ldots, p_3$ and $T_{\overline{\mathbf{k}}}$ contains precisely six (-1)-curves:

the three exceptional divisors and the three strict transform of the lines through two of the p_i.

The case $(\mathbb{P}^2, 1, 2)$:
Let $T \longrightarrow \mathbb{P}^2$ be the blow-up of a rational point and a point of degree 2, and denote by E the exceptional divisor of the rational point and by E'_1, E'_2 the geometric components of their exceptional divisor of the point of degree 3. Let $L \subset T$ be the pullback of a general line. The $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ are:

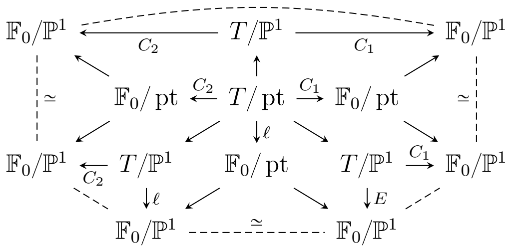
$$E, \quad E'_1, E'_2$$
$$\ell := L - E'_1 - E'_2, \quad \ell'_i := L - E - E'_i$$
$$E \cdot E'_i = E \cdot \ell = \ell'_i \cdot \ell = 0, \ i = 1, 2$$
$$E'_i \cdot \ell'_j = \delta_{ij}$$



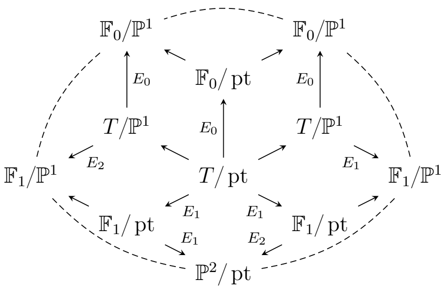
By completing the contraction diagram starting from T, we obtain the above diagram.
The remaining case $(\mathcal{Q}, 1, 1)$ appears in the above diagram.

The case $(\mathbb{F}_0, 4)$:
Let $T \longrightarrow \mathbb{F}_0$ be the blow-up of a point of degree 2, and denote by E_1, E_2 the geometric components of its exceptional divisor. Let F_1, F_2 the the classes of the fibres of the two fibrations of $\mathbb{F}_0$. From Remark A.2.3 we obtain that the only $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$-orbits of (-1)-curves in $T_{\overline{\mathbf{k}}}$ of cardinality $\leqslant 3$ are the following, where the number in parentheses indicates the number of geometric components:

$$E_1, E_2$$
$$C_{ji} := F_j - E_i$$
$$E_i C_{jk} = \delta_{ik}$$



Suppose that T/pt is a rank 3 fibration with $K_T^2 = 7$. Then it has precisely three rational (-1) curves: E_0, E_1, E_2, the contractions of which yield the following commutative diagram, which is the case $(\mathbb{P}^2, 1, 1)$.



This completes the list of elementary relations in $\mathrm{BirMori}(\mathbb{P}^2_{\mathbf{k}})$ for a perfect field $\mathbf{k}$.

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