Bachelor Mathematik, Universität Basel, FS2024

Übungsblatt 9

Hand in until 28 April.

Exercice 1. Describe the set of points where f is complex-differentiable and compute $f'(z)$ for these points.

(1) $f(x+iy) = x^4y^5 + ixy^3$
(2) $f(x+iy) = y^2 \sin x + iy$
(3) $f(x+iy) = \sin^2(x+y) + i\cos^2(x+y)$
(4) $f(x+iy) = -6(\cos x + i \sin x) + (2-2i)y^3 + 15(y^2+2y)$
(5) $f(x+iy) = \frac{1}{2}\log(x^2+y^2) + i \arctan\left(\frac{y}{x}\right)$.

Exercice 2. Show that if $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, then $p'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}$.

Exercice 3. Let $D \subset \mathbb{C}$ be open and $f\colon D \longrightarrow \mathbb{C}$ be real differentiable. Show that $\det Df = |f_z|^2 - |f_{\bar{z}}|^2$.

Exercice 4. Let $D, D' \subset \mathbb{C}$ be open. A map $f\colon D \longrightarrow \mathbb{C}$ is called *biholomorphic* if $f(D)$ is open, f is bijective and $f^{-1}\colon f(D) \longrightarrow \mathbb{C}$ is holomorphic. We denote by $Q_r(c) \subset \mathbb{C}$ the open square of center c and side-length $2r$. We define

$$\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}, \quad \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}, \quad \mathbb{C}^- := \mathbb{C}\backslash\{z \in \mathbb{C} \mid \Re(z) \leqslant 0, \Im(z) = 0\}$$

and

$$h_C\colon \mathbb{C}\backslash\{-i\} \longrightarrow \mathbb{C},\ z \mapsto \frac{z-i}{z+i}, \quad h_{C'}\colon \mathbb{C}\backslash\{1\} \longrightarrow \mathbb{C},\ z \mapsto i\frac{1+z}{1-z}$$

(1) Show that $h_c \circ h_{C'} = h_{C'} \circ h_C = \mathrm{id}$.
(2) Show that $1 - |h_C(z)|^2 = \frac{4\Im(z)}{|z+i|^2}$ for $z \neq -i$ and $\Im(h_{C'}(z)) = \frac{1-|z|^2}{|1-z|^2}$ for $z \neq 1$.
(3) Conclude that $h_C(\mathbb{H}) \subset \mathbb{D}$ and $h_{C'}(\mathbb{D}) \subset \mathbb{H}$.
(4) Conclude that $h_C\colon \mathbb{H} \longrightarrow \mathbb{D}$ is biholomorphic.
(5) Show that the map $\varphi\colon \mathbb{H} \longrightarrow \mathbb{C}^-$, $z \mapsto -z^2$ is holomorphic and bijective.
(6) Show that the map $\psi\colon \mathbb{D} \longrightarrow \mathbb{C}^-$, $z \mapsto \left(\frac{z+1}{z-1}\right)^2$ is holomorphic and bijective.
(7) Show that $h_C\colon \mathbb{H} \longrightarrow \mathbb{D}$ maps $B_1(0)$ biholomorphically onto $\{w \in \mathbb{D} \mid \Im(w) < 0\}$.

Exercice 5. Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$, $z \mapsto \frac{az+b}{cz+d}$ with $c \neq 0$ and $ad - bc \neq 0$. Let $L \subset \mathbb{C}$ be a circle or a real line. Determine the images $f(L)$ and of $f(L\backslash\{-d/c\})$.
Hint: write $f(z) = \frac{bc-ad}{c^2}(z+d/c)^{-1} + \frac{a}{c}$.

Exercice 6. Suppose $S \subset M$ is a level set of a smooth submersion $\Phi = (\phi_1, \ldots, \phi_k) : M \to \mathbb{R}^k$. Show that an element $v \in T_pM$ is tangent to S if and only if $v(\phi_1) = \cdots = v(\phi_k) = 0$.
Hint: will be given in the exercise lecture on 22 April.

Exercice 7. For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by

$$M_a = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; y^2 = x(x-1)(x-a)\right\}.$$

(1) For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$?

(2) For which values of a can M_a be given a topology and smooth structure making it into an immersed submanifold?

. *Hint: will be given in the exercise lecture on 22 April.*