Bachelor Mathematik, Universität Basel, FS2024

Übungsblatt 8

Exercice 1. Consider the map $\phi : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$, $(x, y, s, t) \mapsto \big(x^2 + y,\, x^2 + y^2 + s^2 + t^2 + y\big)$. Show that $(0, 1)$ is a regular value of ϕ, and that the level set $\phi^{-1}(0, 1)$ is diffeomorphic to $\mathbb{S}^2$.

Exercice 2. Show that the image of the curve $\beta : (-\pi, \pi) \longrightarrow \mathbb{R}^2$, $t \mapsto (\sin 2t, \sin t)$ from Example 5.20(4) is not an embedded submanifold of $\mathbb{R}^2$.
Careful: this is not the same as showing that β is not an embedding.

Exercice 3. Let M be a smooth manifold and $U \subset M$ an open subset. Show that the smooth structure on U inherited from M makes U a (smooth) submanifold of M.

Exercice 4. Suppose M is a smooth manifold and $S \subset M$ is an embedded submanifold. Then S is properly embedded if and only if it is a closed subset of M.

Exercice 5. Let M, B be smooth manifolds and $f : M \longrightarrow N$ is a smooth map. Consider the graph $\Gamma(f) := \{(x, f(x)) \in M \times N \mid x \in M\}$, of f endowed with the smooth manifold structure from Exercise sheet 7. Show that $\Gamma(f)$ is properly embedded in $M \times N$.

Exercice 6. Let M and N be smooth manifolds and $\phi : M \longrightarrow N$ a smooth map with constant rank r. Show that each level set L_ϕ of ϕ is a properly embedded submanifold of M of dimension $\dim L_\phi = \dim M - r$.

Exercice 7. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \mapsto x^3 + xy + y^3$. For each level set, show either that it is or that it is not an embedded submanifold.