Übungsblatt 7

Exercice 1. Show that the set of $m \times n$ matrices of full rank is an open subset of $M_{mn}(\mathbb{R})$.

Exercice 2. We define the complex projective space as $\mathbb{CP}^n = (\mathbb{C}^{n+1} \backslash \{0\})/\sim$, where $x \sim y$ if there exists $\lambda \in \mathbb{C}^*$ such that $y = \lambda x$. The class of $x \in \mathbb{C}^{n+1} \backslash \{0\}$ is denoted by $[x]$.

(1) For $i = 0, \ldots, n$, let $U_i := \{[x] \in \mathbb{CP}^n \mid x_i \neq 0\}$ and $\varphi_i \colon U_i \longrightarrow \mathbb{C}^n$, $[x] \mapsto (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Show that (U_i, φ_i) is a chart of $\mathbb{CP}^n$ and that the family $\{(U_i, \varphi_i)\}_{i=0}^n$ provides a smooth atlas of $\mathbb{CP}^n$.

(2) Show that the quotient map $\mathbb{C}^{n+1} \backslash \{0\} \longrightarrow \mathbb{CP}^n$, $x \mapsto [x]$, is a surjective smooth submersion.

(3) Show that $\mathbb{CP}^1 \approx \mathbb{S}^1$.

Exercice 3. Let M be a smooth compact manifold. Show that there is no smooth submersion $F \colon M \longrightarrow \mathbb{R}^k$ for any $k \geqslant 1$.

Exercice 4. Suppose M, N are smooth manifolds, $U \subset M$ is open and $f : U \longrightarrow N$ is a smooth map. Let $\Gamma(f) \subset M \times N$ denote the graph of f:

$$\Gamma(f) = \{(x, y) \in M \times N \mid x \in U, y = f(x)\}.$$

Show that $\Gamma(f)$ is an embedded submanifold of $M \times N$.