Übungsblatt 5

Exercice 1. Let M be a smooth manifold and (U, φ) a smooth chart of M. Show that $\varphi\colon U \longrightarrow \varphi(U)$ is a diffeomorphism.

Exercice 2. Let $M_1, \ldots, M_k$ be smooth manifolds and let $\pi_j : M_1 \times \cdots \times M_k \longrightarrow M_j$ be the projection onto M_j. For each point $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, we define a map by

$$\alpha : T_p(M_1 \times \cdots \times M_k) \longrightarrow T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k,\ \ \alpha(v) = \big(d\pi_1(p)(v), \ldots, d\pi_k(p)(v)\big).$$

Show that it is an isomorphism of vector spaces.

Exercice 3. Suppose M and N are smooth manifolds and let $F\colon M \longrightarrow N$ be a smooth map. Show that $DF_p : T_pM \longrightarrow T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each connected component of M.

Exercice 4. Let $F : M \longrightarrow N$ be a smooth map between smooth manifolds, and let $\gamma : J \longrightarrow M$ be a smooth curve. Show that for any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma : J \longrightarrow N$ is given by

$$(F \circ \gamma)'(t_0) = DF_{\gamma(t_0)}(\gamma'(t_0)).$$

Exercice 5. Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ under the usual identification $\mathbb{C}^2 \cong \mathbb{R}^4$. For each $z = (z_1, z_2) \in \mathbb{S}^3$, define a curve $\gamma_z : \mathbb{R} \longrightarrow \mathbb{S}^3$ by

$$\gamma_z(t) = (e^{it}z_1, e^{it}z_2).$$

Show that $\gamma_z(t)$ is a smooth curve whose velocity is never zero.