Übungsblatt 4

To hand in on Monday, 17 March 2025.

Exercice 1. Let $\mathbb{S}^n \subset \mathbb{R}^n$ be the unit sphere and consider the map $f\colon \mathbb{S}^n \longrightarrow \mathbb{RP}^n$, $p \mapsto [p]$. Show that f is smooth.

Exercice 2. Let $P\colon \mathbb{R}^{n+1}\backslash\{0\} \longrightarrow \mathbb{R}^{k+1}\backslash\{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$ we have $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R}^*$ and for all $x \in \mathbb{R}^{n+1}\backslash\{0\}$. *(Such a function is called homogeneous of degree d.)* We denote by $[y]$ the class of $y \in \mathbb{R}^{n+1}\backslash\{0\}$ in $\mathbb{RP}^n$. Show that the map

$$\tilde{P} : \mathbb{RP}^n \longrightarrow \mathbb{RP}^k, \quad \tilde{P}([x]) = [P(x)]$$

is well-defined and smooth.

Exercice 3. Let M be a nonempty smooth n-manifold with or without boundary, and suppose $n \geqslant 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional.
Hint: Show that if $f_1, \ldots, f_k$ are elements of $C^\infty(M)$ with non-empty disjoint supports, then they are linearly independent.

Exercice 4. Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists a function $f \in C^\infty(M)$ such that $0 \leqslant f(x) \leqslant 1$ for all $x \in M$ and $f^{-1}(0) = A$ and $f^{-1}(1) = B$.
Hint: Use a theorem from the lecture and a partition of unity.

Exercice 5. Let $M_1, \ldots, M_k$ be smooth manifolds and let $\pi_j : M_1 \times \cdots \times M_k \longrightarrow M_j$ be the projection onto M_j. For each point $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, we define a map by

$$\alpha : T_p(M_1 \times \cdots \times M_k) \longrightarrow T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k,\ \alpha(v) = \big(d\pi_1(p)(v), \ldots, d\pi_k(p)(v)\big).$$

Show that it is an isomorphism.