Übungsblatt 2

Exercice 1. (1) Show that any finite dimensional vector space is a topological manifold.

(2) Show that $M_n(\mathbb{R})$ und $GL_n(\mathbb{R})$ are topological manifolds for any $n \geqslant 1$.

(3) For $n \in \{1, 2\}$, let $\mathbb{S}^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$. Show that $\mathbb{S}^n$ is an n-dimensional topological manifold. Use the stereographic projection $\mathbb{S}^n \backslash \{(0, \ldots, 0, 1)\} \longrightarrow \mathbb{R}^n$, $x \mapsto \frac{1}{1-x_{n-1}}(x_1, \ldots, x_n)$, whose inverse is

$$y \mapsto \frac{1}{|y|^2 + 1}(2y_1, \ldots, 2y_n, |y|^2 - 1)$$

(4) Let $n \geqslant 1$. We define the real projective space $\mathbb{R}^n\mathbb{P}^n := \mathbb{R}^{n+1} \backslash \{0\} / \sim$, where $x \sim y$ if there exists $\lambda \in \mathbb{R}^*$ such that $y = \lambda x$. Show that it is a topological manifold.

Exercice 2. Let $U \subset \mathbb{R}^n$ be an open set with its standard smooth structure and let $f \colon U \longrightarrow \mathbb{R}^k$ be a map. Show that f is smooth in the sense of the course if and only if it is smooth in the sense of Analysis I&II.

Exercice 3. Let $U \subset \mathbb{R}^n$ be an open set and $f \colon U \longrightarrow \mathbb{R}$ be a smooth function. Let $c \in \mathbb{R}$ and consider the set $f^{-1}(c)$ (called level set of f). Suppose that $Df_a \neq 0$ for any $a \in f^{-1}(c)$. Show that $f^{-1}(c)$ has a smooth structure. (Hint: use the implicit function theorem from Analysis II.)