Übungsblatt 13

Exercice 1. Let $D \subset \mathbb{C}$ be open and bounded and $f, g\colon \overline{D} \longrightarrow \mathbb{C}$ continuous and without zeros. Suppose that $f, g\colon D \longrightarrow \mathbb{C}$ are holomorphic and $|f(z)| = |g(z)|$ for all $z \in \partial D$. Show that there exists $\lambda \in \mathbb{S}^1$ such that $f(z) = \lambda g(z)$ for all $z \in \overline{D}$.

Exercice 2. Let $p \in \mathbb{C}[z]$ be a non-constant polynomial. Show that $p(\mathbb{C}) = \mathbb{C}$.

Exercice 3.

(1) Let $p \in \mathbb{C}[z]$ of degree $\deg(p) = n$ and let $c_1, \ldots, c_n \in \mathbb{C}$ be its (not necessarily distinct) zeros. Show that $\frac{p'(z)}{p(z)} = \sum_{j=1}^n \frac{1}{z-c_j}$.

(2) Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be holomorphic and suppose it has only finitely many zeros $c_1, \ldots, c_n$. Show that $\frac{f'(z)}{f(z)} = \sum_{j=1}^n \frac{1}{z-c_j}$ for all $z \in \mathbb{C} \backslash \{c_1, \ldots, c_n\}$ if and only if $f \in \mathbb{C}[z]$.

We assume the following theorem:

Theorem 0.1 (Riemann-Hurwitz)**.** *Let $f\colon X \longrightarrow Y$ be a non-constant holomorphic map between compact connected orientable Riemann surfaces. Then*
$2g(X) - 2 = d(2g(Y) - 2) + \sum_{p \in X}(\mathrm{ord}_p f - 1)$*, where* $d = \sum_{p \in f^{-1}(q)} \mathrm{ord}_p f$ *for any* $q \in Y$*.*

Exercice 4. Let $f\colon X \longrightarrow Y$ be a non-constant holomorphic map between compact connected orientable Riemann surfaces.

(1) Suppose that $X = \mathbb{P}^1_{\mathbb{C}}$. Show that Y is biholomorphic to $\mathbb{P}^1_{\mathbb{C}}$.
(2) Suppose that $g(X) = 1$. Show that Y diffeomorphic to a torus or to a sphere.
(3) Suppose that $g(X) = g(Y) = 1$. Show that f is a finite covering (without ramification).

Exercice 5.

(1) Draw an example for the Riemann-Hurwitz theorem for each $d \geqslant 1$.
(2) Draw an example for the Riemann-Hurwitz theorem with $g(Y) \geqslant 2$.