Übungsblatt 12

Exercice 1. Let $z \in \mathbb{C}^- := \mathbb{C}\backslash\mathbb{R}_{\leqslant 0}$ and γ a path from 1 to $z = r^{i\varphi}$. Compute that $\int_\gamma \frac{d\zeta}{\zeta} = \log(r) + i\varphi$.

Exercice 2. Let $D \subset \mathbb{C}$ be open and $f\colon D \longrightarrow \mathbb{C}$ a holomorphic map. Suppose that $f\colon D \longrightarrow f(D)$ is bijective. Show that f^{-1} is holomorphic.

Exercice 3. We define

$$J := \left\{ z \mapsto e^{i\varphi}\frac{z-w}{\bar{w}z-1} \mid w \in B_1(0),\ 0 \leqslant \varphi < 2\pi \right\}, \qquad M := \left\{ B := \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \mid \det B = 1 \right\},$$

$$N := \left\{ B := \begin{pmatrix} \eta & -\eta w \\ \bar{w}z & -1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \mid \det B = 1 \right\}$$

(1) Show that for any $f, g \in J$, we have $f \circ g \in J$.
(2) Show that any $f \in J$ is holomorphic, has an inverse, and f^{-1} is also holomorphic.
(3) Show that $\{f \in J \mid f(0) = 0\} = \left\{ z \mapsto -ze^{i\varphi} \mid 0 \leqslant \varphi < 2\pi \right\}$.
(4) Show that there exists $a \in \mathbb{C}^*$ such that $a^2 = \frac{-\eta}{1-|w|^2}$
(5) Show that $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in B$.
(6) Show that for $s := \frac{\eta}{a} \in \mathbb{C}^*$, we have $W = sB$.

We assume the following three statements:

Theorem (Open Mapping theorem)**.** *Let $D \subset \mathbb{C}$ be open and $f\colon D \longrightarrow \mathbb{C}$ holomorphic. If f nowhere locally constant on D, then f is open.*

Theorem (Liouville)**.** *Every bounded holomorphic map $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ is constant.*

Lemma (Growth Lemma)**.** *Let $p(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}[z]$ be a polynom of degree $n \geqslant 0$. Then there is $R > 0$, such that for all $z \in \mathbb{C}$ with $|z| \geqslant R$, we have $\frac{1}{2}|a_n||z|^n \leqslant |p(z)| \leqslant \frac{3}{2}|a_n||z|^n$ and $\lim_{|z| \longrightarrow \infty} \left| \frac{z^k}{p(z)} \right| = 0$ for $0 \leqslant k < n$*

Exercice 4. Show that every holomorphic map $f\colon \mathbb{C} \longrightarrow B_1(0)$ is constant.
Conclude that there are no biholomorphic mapping of $\mathbb{H} \longrightarrow \mathbb{C}$ (where $\mathbb{H} \subset \mathbb{R}^2$ is the open upper halfplane).

Exercice 5. Show that any polynomial $p \in \mathbb{C}[t]$ has a zero in $\mathbb{C}$.
Hint: use Liouville's theorem and the Growth lemma

Exercice 6. Let $D \subset \mathbb{C}$ be a connected open set and $f\colon D \longrightarrow \mathbb{C}$ a holomorphic map. Suppose f has a local maximum at the point $c \in D$ (i.e. $|f(c)| = \|f\|_U$ for some neighbourhood $U \subset D$ of c). Show that f is constant on D.

Exercice 7. Let $D \subset \mathbb{C}$ be open and connected and $f\colon D \longrightarrow \mathbb{C}$ holomorphic. Let $U \subset D$ be open and assume that $\max_{z \in U} |f(z)|$ exists. Show that f is constant on D. *Hint: use the Open Mapping Theorem*