**Bachelor Mathematik, Universität Basel, FS2024**

# Übungsblatt 10

**Exercice 1.** Let $n \in \mathbb{Z}$, $r > 0$ and $B_r(c) = \{z \in \mathbb{C} \mid |z - c| < r\} \subset \mathbb{C}$. Show that

$$\int_{\partial B_{r(c)}} (\zeta - c)^n d\zeta = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$

*Hint: parametrize $\partial B_r(c)$ smartly.*

**Exercice 2.** Show that

$$\int_{\partial B_r(c)} \frac{1}{\zeta - z} d\zeta = \begin{cases} 1, & z \in B_r(c) \\ 0, & z \notin B_r(c) \end{cases}$$

**Exercice 3.** Let $r, s > 0$ and consider the rectangle $R = \{z \in \mathbb{C} \mid -r < \Re(z) < r,\ -s < \Im(z) < s\}$. Compute

$$\int_{\partial R} \frac{1}{\zeta} d\zeta.$$

**Exercice 4.** Let $D \subset \mathbb{C}$ be open and $f\colon D \longrightarrow \mathbb{C}$ continuous. Suppose that there is a holomorphic map $F\colon D \longrightarrow \mathbb{C}$ such that $F' = f$. Show that for every pair $w, z \in D$ and every (piecewise continuous differentiable) path $\gamma$ in $D$ with initial point $w$ and endpoint $z$, we have

$$\int_\gamma f d\zeta = F(z) - F(w).$$

**Exercice 5** (Integrability Criterion)**.** Let $D \subset \mathbb{C}$ be open and $f\colon D \longrightarrow \mathbb{C}$ continuous. Show that the two following assertions are equivalent:

(1) there exists a complex differentiable map $F\colon D \longrightarrow \mathbb{C}$ such that $F' = f$;
(2) $\int_\gamma f d\zeta = 0$ for every closed (piecwise differntiable) path $\gamma$ in $D$.

*Hint 1: Suppose that (2) holds and that $D$ is connected. Fix $z_0 \in D$. For every $z \in D$, let $\gamma_z$ be a (piecwise differntiable path) from $z_1$ to $z$ in $D$. Set $F(z) := \int_{\gamma_z} f d\zeta$, $z \in D$.*
*Hint 2: use exercise 4.*