Bachelor Mathematik, Universität Basel, FS2024

Übungsblatt 1

Exercise 0.1. Let $(X, \tau), (Y, \tau_Y), (Z, \tau_Z)$ be topological spaces. Show that $(X \times Y) \times Z \simeq X \times (Y \times Z)$.

Exercise 0.2. We endow $\mathbb{R}$ with its usual Euclidean topology. Show that the product topology on $\mathbb{R}^n$ is the Euclidean topology.

Exercise 0.3. Show that $\mathrm{GL_n}(\mathbb{R}) = \{\det \neq 0\} \subset \mathrm{M_n}(\mathbb{R})$ is open in the Euclidean Topology.

Exercise 0.4. Let X be a topological space and $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ a locally finite family of subsets of X. Show that $\{\overline{X_\alpha}\}_{\alpha \in A}$ is locally finite and that $\overline{\cup_\alpha X_\alpha} = \cup_\alpha \overline{X_\alpha}$.

Exercise 0.5. (Problem 1.1 in Lee) Define the line with two origins (I will not do it in the course). Show that it is non-Hausdorff, and that is locally Euclidean and second-countable.

Examples of Manifolds:

Exercise 0.6. Show that the sphere $\mathbb{S}^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$ is an n-dimensional smooth manifold (using the stereographic projection). Use the stereographic projection $\mathbb{S}^n \backslash \{(0, \ldots, 0, 1)\} \longrightarrow \mathbb{R}^n$, $x \mapsto \frac{1}{1 - x_{n-1}}(x_1, \ldots, x_n)$, whose inverse is

$$y \mapsto \frac{1}{|y|^2 + 1}(2y_1, \ldots, 2y_n, |y|^2 - 1)$$

Exercise 0.7. Show that any finite dimensional linear vector space is a smooth manifold. In particular, $M_n(\mathbb{R})$ and $\mathrm{GL_n}(\mathbb{R})$ are smooth manifolds.

Exercise 0.8. Define the real projective space $\mathbb{R}^n\mathbb{P}^n := \mathbb{R}^{n+1} \backslash \{0\} / \sim$, where $x \sim y$ if there exists $\lambda \in \mathbb{R}^*$ such that $y = \lambda x$. Show that it is a smooth manifold.